Scaling properties of multifractal functions at an attractor-repeller transition

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Multifractal properties of repelling sets generated by hyperbolic maps are studied as a function of a parameter describing a transition to an attracting interval. Critical indices in the scaling behavior of multifractal functions are found when a uniform probability density is assumed. A constant probability is also considered, and the resulting thermodynamic-like functions are investigated close to the critical value of the parameter.

I. INTRODUCTION

It has recently been established that fractal sets appearing in nonlinear systems and supporting a probability distribution can be characterized in general by a spectrum of scaling indices. This basis of this multifractal analysis lies on a coverage of the set by $N$ disjoint pieces of radii $\{l_i\}_{i=1}^N$, each having a probability $p_i$, in order to construct a partition function:

$$\Gamma(q, \tau, \{l_i\}) = \sum_{i=1}^N \frac{p_i^q}{l_i^\tau}. \quad (1)$$

It has been argued that when $N \to \infty$, $\Gamma$ tends to infinity for $\tau > \tau(q)$ and to zero for $\tau < \tau(q)$. Usually, $\tau(q)$ is calculated by requiring that $\lim_{N \to \infty} \Gamma(q, \tau, \{l_i\}) = 1$. The function $\tau(q)$ is related to the generalized dimensions $D_q$, $\tau(q) = (q-1)D_q$. A Legendre transform of $\tau(q)$ yields the $f(\alpha)$ function, another convenient representation of the scaling properties of fractal measures. This formalism has been applied to both theoretical and experimental studies of chaotic attractors in dynamical systems, dissipation fields of fully developed turbulence, diffusion-limited aggregates, nonlinear resistor networks, etc. Because the various quantities derived from the partition function describe global properties of the systems, they have been compared with thermodynamic functions. Nonanalytic behaviors in these functions have been interpreted as phase transitions.

The multifractal functions depend on the parameters defining the partition $\{l_i\}_{i=1}^N$. For example, a small deviation of a parameter from its value at the onset of chaos via the period-doubling or the quasiperiodic routes affects the observable $f(\alpha)$ spectra of the corresponding attractors. The knowledge of these kinds of effects is important in an experimental situation since $f(\alpha)$ or $D_q$ can be used to characterize a particular transition to chaos. Besides attractors, however, there exist repellers, which are invariant fractal sets related to transient behavior in dissipative dynamical systems. Some chaotic maps have attractors at some critical parameter value above which the attractor becomes unstable and gives place to a repeller, a situation called a boundary crisis. In this paper we examine how the thermodynamic multifractal functions scale with a deviation from the critical value of a parameter at this attractor-repeller transition. We investigate the consequences of taking different probability distributions on the geometric support of repellers generated by one-dimensional, one-parameter maps defined in Sec. II; specifically, we consider Gibbs measures of the form $p_i = l_i^\alpha / \sum_{i=1}^N l_i^\alpha$. Two cases are studied. In Sec. II, the measure with $\sigma = 1$, which occurs in some physical repelling sets, is used to analyze $D_q$ as a function of the transition parameter. A scaling function depending only on the order of the map characterizes the deviation of any dimension $D_q$ from its value at criticality, and it is calculated for some cases. A balanced measure, corresponding to $\sigma = 0$, is treated in Sec. III. This measure is also interesting since it induces nonanalyticities in the thermodynamic functions. Scaling properties near critical points are also found.

II. UNIFORM PROBABILITY DENSITY

Consider the family of hyperbolic maps $f_\epsilon: [0,1] \to [0,1]$ given by

$$f_\epsilon = \left(1 + \epsilon \right) |1 - |2\pi x| |, \quad (2)$$

with $\epsilon \geq 0$, $x > 1$. $\epsilon$ is the distance from the chaotic map $f_0$ which fully maps the unit interval onto itself. $x$ is the order of the map at the critical point $x_c = \frac{1}{2}$, $f_0(x_c) = 1$. Then $f_\epsilon^{(-n)}([0,1]) \subset [0,1]$ and $|f_\epsilon^{(n)}(x)| > 1$ on $x \in f_\epsilon^{(-n)}([0,1])$ for some $n \geq 1$. Because of the expanding property of the map, almost all points will eventually escape the interval $[0,1]$. There is, however, an invariant Cantor set, the repeller $f = \cap_n \cup f_\epsilon^{(-n)}([0,1])$, which remains in $[0,1]$. The remaining set $J_\epsilon$ after $n$ iterations of $f_\epsilon$ is the union of the intervals $l_i^{(n)}, i = 1, \ldots, 2^n$, obtained as

$$l_i^{(n)} = \left| \tilde{e}_i^{(n)} - e_i^{(n)} \right|, \quad (3)$$

where the points $\tilde{e}_i^{(n)}$ and $e_i^{(n)}$ satisfy

$$f_\epsilon^{(n)}(\tilde{e}_i^{(n)}) = 1, \quad f_\epsilon^{(n)}(e_i^{(n)}) = 0, \quad (4)$$

and they can be found recursively. The fixed points of $f_\epsilon^{(n)}$ are defined by

$$f_\epsilon^{(n)}(x) = x$$

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\[ f^{(n)}_\varepsilon(x_i) = x_i^{(n)}, \quad i = 1, \ldots, 2^n. \] (5)

In particular, \( x_i^{(n)} = 0 \), and \( x_i^{(1)} = x_i^{(n)} \) is the solution of \( (2x_i^{(1)} - 1)^2 = 1 - \frac{x_i^{(1)}}{1+\varepsilon} \). (6)

The points \( x_i^{(n)} \) belong to \( J_n \), and each interval \( I_i^{(n)} \) contains precisely one fixed point of \( f^{(n)}_\varepsilon \). Figure 1 presents \( f^{(3)}_\varepsilon \) for the quadratic case \( (z=2) \), and the sets \( J_n \) are shown for successive \( n \). By the mean value theorem there exists an \( x_i \in I_i^{(n)} \), such that \( I_i^{(n)} = |f^{(n)}_\varepsilon(x_i)|^{-1} \). Since \( I_i^{(n)} \) decreases exponentially with \( n \), and the principle of bounded variation\(^1\) ensures that growth rates are unchanged if we choose different points in \( I_i^{(n)} \), we must have

\[ \lim_{n \to \infty} \left[ |f^{(n)}_\varepsilon(x_i)| - |f^{(n)}_\varepsilon(x_i^{(n)})| \right] = 0. \]

Then we can write in the limit of large \( n \):

\[ I_i^{(n)} = |f^{(n)}_\varepsilon(x_i^{(n)})|^{-1}. \] (7)

At large \( n \), the smallest and the largest intervals are\(^{11}\)

\[ I_{\text{min}} = |f^{(n)}_\varepsilon(0)|^{-1} = |2z(1+\varepsilon)|^{-n}, \]

\[ I_{\text{max}} = |f^{(n)}_\varepsilon(x_i^{(1)})|^{-1} = |2z(1+\varepsilon)|2x_i^{(1)} - 1|^{-n} \]. (9)

If we start with a large number of points uniformly distributed over the unit interval and associate the probability \( p_i^{(n)} \) with the number of points remaining in the interval \( I_i^{(n)} \) after \( n \) iterations, we have \( p_i^{(n)} \propto I_i^{(n)} \). This corresponds to a Gibbs measure with \( \sigma = 1 \), or geometric multifractality.\(^{18}\) As a recent example,\(^{20}\) the repelling sets associated with the onset of chaotic scattering have this property. Distributions of constant probability density also occur for growth processes.\(^{18}\) The partition function [Eq. (1)] in this case becomes

\[ \Gamma_n = \frac{\sum_{i=1}^{2^n} (l_i^{(n)})^{q-D_q(q-1)}}{\left(\sum_{i=1}^{2^n} l_i^{(n)}\right)^q}, \] (10)

where the denominator can be related to the escape rate of the repeller\(^{11}\):

\[ r = -\lim_{n \to \infty} \frac{1}{n} \ln \left( \sum_{i=1}^{2^n} l_i^{(n)} \right), \]

allowing, for \( n \to \infty \), the expression

\[ -nqr = \ln \left( \sum_{i=1}^{2^n} (l_i^{(n)})^{q-D_q(q-1)} \right). \] (12)

Since \( l_i^{(n)} \) is a function of \( \varepsilon \), Eq. (12) determines \( D_q(\varepsilon) \). Because \( \sum_{i=1}^{2^n} l_i^{(n)}(\varepsilon=0) = 1 \), we obtain \( D_q(0) = 1 \) for all \( q \).

In order to investigate the scaling properties of \( D_q(\varepsilon) \) when \( \varepsilon \to 0 \), consider first \( q \to +\infty \). Then \( I_i^{(n)} \) maximizes the sum in Eq. (12), and we get

\[ 1 - D_{+\infty}(\varepsilon) = -\frac{nr}{\ln l_{\text{max}}} = \frac{r}{\ln[2z(1+\varepsilon)] - 1} \]. (13)

Similarly, for \( q \to -\infty \), only \( l_i^{(n)} \) survives in the sum

\[ 1 - D_{-\infty}(\varepsilon) = -\frac{nr}{\ln l_{\text{min}}} = \ln[2z(1+\varepsilon)]. \] (14)

Differentiating Eq. (12) with respect to \( q \) and setting \( q = 1 \), we also get

\[ 1 - D_1(\varepsilon) = -\frac{n}{\sum_{i=1}^{2^n} l_i^{(n)}} \sum_{i=1}^{2^n} l_i^{(n)} \ln l_i^{(n)} = \frac{nr}{\sum_{i=1}^{2^n} l_i^{(n)}}, \] (15)

where the coefficient of \( r \) can be identified with the inverse of the effective Lyapunov exponent.\(^{16}\) For general \( q \), we define \( s \equiv 1 - D_q(\varepsilon) \) and \( \phi_q(s) \equiv \sum l_i^{(n)} 1^{s-1} \). Then the limit \( \varepsilon \to 0 \) is equivalent to \( s \to 0 \). Equation (12) can be expressed as

\[ -nqr = \ln \phi_q(s) \]. (16)

A Taylor expansion of \( \phi_q(s) \) about \( s = 0 \) gives

FIG. 1. The third iterate of \( f_\varepsilon \) for \( z = 2 \) and \( \varepsilon = 0.5 \). The intervals \( I_i^{(n)} \) corresponding to \( f_\varepsilon^{(n)} \) are shown for \( n = 1, 2, 3 \). At each level \( n \), the smallest \( l_i^{(n)} \) and the largest \( l_i^{(n)} \) intervals are indicated. The fixed points \( x_i^{(1)} = 0 \) and \( x_i^{(1)} \) always belong to the smallest and largest intervals of each level, respectively.
\[ -nq = \ln \phi_q(0) + \frac{d \ln \phi_q(s)}{ds} \bigg|_{s=0} + \frac{d^2 \ln \phi_q(s)}{ds^2} \bigg|_{s=0} s^2 + \cdots . \]  

The escape rate has been shown to possess a universal scaling as \( r = \rho(x_c) \epsilon^{1/2} \), for \( \epsilon \to 0 \), where \( \rho(x_c) \) is the invariant measure density for \( f_0 \) at the critical point \( x_c \), with \( \rho(x) \propto [x(1-x)]^{1/2-1} \). Therefore, in the limit \( \epsilon \to 0 \), we can obtain the following from Eq. (17):

\[ 1 - D_q(\epsilon) = \rho(x_c) A_q(z) \epsilon^{1/2}, \]  

where

\[ A_q(z) = \lim_{\epsilon \to 0} - \frac{n}{\sum_{i=1}^{2^n} (I_{n(i)}^q - D_q^{(q-1)}) \ln |I_{n(i)}|} \]  

is a scaling function for each \( q \) depending only on the order of the maximum of symmetric maps on the unit interval. In particular, \( A_{+\infty}(2) = (\ln 4)^{-1} \), \( A_{+\infty}(2) = (\ln 2)^{-1} \).

We have confirmed the scaling [Eq. (18)] numerically. In practice, in order to improve convergence, we take the ratio

\[ \frac{\Gamma_{n-1}}{\Gamma_n} = 1, \]  

with \( n = 15 \), and obtain \( D_q \) for fixed \( q \) and varying \( \epsilon \). Figure 2 shows \( 1 - D_q(\epsilon) \), obtained from the quadratic case, versus \( \epsilon^{1/2} \), for several values of \( q \) and \( \epsilon < 10^{-3} \). The slopes of the lines give \( \rho(x_c) A_q(2) \). The constant \( \rho(x_c) \)

was estimated numerically in general from the slope of \( r \) versus \( \epsilon^{1/2} \). Figure 3 shows the scaling functions \( A_q(z) \) corresponding to \( z = 2 \) and \( 3 \). The curves \( A_q(z) \) present two asymptotes in the limit \( q \to \pm \infty \) which are likely to be \( A_{+\infty}(z) \) and \( A_{-\infty}(z) \). The ratio \( A_{+\infty}(z)/A_{-\infty}(z) \) \( (q > 0) \) is always greater than 1 and tends to the limit

\[ A_{+\infty}(z) = \lim_{\epsilon \to 0} \ln \frac{\ln |I_{n(\max)}|}{\epsilon \ln |I_{n(\min)}|} = \frac{\ln(2z)}{\ln(2z) + (z-1)\ln[2x_{1/2}(\epsilon = 0) - 1]}, \]  

which depends only on \( z \) and characterizes the asymmetry of the relative shifts of the generalized dimensions.

Table I shows this ratio for several values of \( z \). The dimensions \( D_q(\epsilon) \) with \( q > 0 \) are more sensitive to a deviation from homogeneity than those with \( q < 0 \). The deviation increases as \( q \to +\infty \), corresponding to the largest scale of the repeller. The parameter \( \epsilon \) describes a transition between simple fractal behavior \( (D_q = \text{const}) \) and multifractality, with critical index 1/2. The function \( A_q(z) \) measures the sensitivity of each \( D_q \) to this transition.

The scaling in the opposite limit, \( \epsilon \to -\infty \), can also be found. In that case the map [Eq. (2)] can be approximat-

![FIG. 2. The differences 1 - D_q(\epsilon) computed from Eq. (20) in the case z = 2 are plotted as a function of \( \epsilon^{1/2} \). The number on each curve indicates the value of q.](image)

![FIG. 3. The scaling functions A_q(z) for z = 2 (circles) and z = 3 (squares) vs q. A_q(2) is obtained from the slopes of the curves in Fig. 2.](image)

**TABLE I.** The ratio of the asymptotic values of the scaling function \( A_q(z) \) calculated from Eqs. (6) and (21). \( z \) is the order of the hyperbolic map which generates the repeller.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( A_{+\infty}(z) )</th>
<th>( A_{-\infty}(z) )</th>
</tr>
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<tr>
<td>1.5</td>
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<td>2.435575</td>
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<td></td>
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</tbody>
</table>
ed by a tentlike map with slopes $\sim 2\varepsilon$. The intervals become equal, and the repeller is a uniform Cantor set. Equation (12) then yields

$$D_q(\varepsilon) = \frac{\ln 2}{\ln(2\varepsilon)}, \quad \varepsilon \to \infty.$$  

(22)

### III. BALANCED MEASURE

A Gibbs measure with $\sigma = 0$ assigns equal probability $p_i^{(n)} = 2^{-n}$ to each of the intervals of the repeller. The partition function for $n \gg 1$ then becomes

$$\Gamma_n = 2^{-nq} \sum_{i=1}^{2^n} (l_i^{(n)})^{-\tau} = 1.$$  

(23)

Such equimeasure partitions occur, for example, in attractors at the onset of chaos. A balanced measure is also generated by the map [Eq. (2)] for general initial conditions. In this situation, a correspondence can be established with an alternative formulation based on a free energy $G(\beta, \varepsilon)$

$$G(\beta, \varepsilon) = -\frac{1}{n} \ln \left( \sum_{i=1}^{2^n} e^{\beta \ln l_i} \right),$$  

(24)

provided that the following identifications are made:

$$\beta = -\tau, \quad G(\beta) = -q(\varepsilon)\ln 2, \quad D_q = \frac{\beta \ln 2}{G(\beta) + \ln 2}.$$  

(25)

$G(\beta)$ can be introduced without any reference to $q$ or $D_q$. Only in the case of sets with equal probability $G(\beta)$ is related to those functions via Eq. (25). To each length scale there can be associated a microscopic energy $E_i(\varepsilon) = |\ln l_i^{(n)}|$, which has its minimum value at $\varepsilon = 0$. For small $\varepsilon$, $E_i(\varepsilon) = |\ln l_i^{(n)}(0)| \sim \varepsilon$. The parameter $\varepsilon$ can be interpreted as an external field that increases the ground energy of the states of the system. We are interested in the scaling of $G(\beta, \varepsilon)$ and $D_q(\varepsilon)$ for the family [Eq. (2)] when $\varepsilon \to 0$. For $\varepsilon = 0$, the map $f_0(y)$ can be conjugated to the tent map $h(y) = 1 - |1 - 2y|, \ y \in [0,1]$; through the change of variables$^{22}$ $y = \int_0^{y'} \rho(x') \, dx'$. However, the case $z = 2$ can be treated directly, since, then,

$$l_i^{(n)}(\varepsilon=0) = |f_0^{(n)}(x_i^{(n)})|^{-1/2} = 2^{-n},$$

except at $x_i^{(n)} = 0$ where $l_i^{(n)}(0) = l_i^{(n)}(0) = 2^{-n}$. Then Eq. (24) gives

$$G(\beta, 0) = \begin{cases} 2\beta \ln 2, & \beta < -1 \\ (\beta - 1) \ln 2, & \beta > -1 \end{cases}.$$  

(26)

Such nonanalyticities in the thermodynamic functions of multifractals have been related with phase transitions.$^{10,11,13}$ The system undergoes a first-order phase transition at $\beta_c = -1$. In the "disordered" phase $\beta > -1$, almost all intervals contribute equally. In the "condensed" phase $\beta < -1$, only the intervals adjacent to the edges contribute to the free energy. Figure 4 shows $G(\beta, \varepsilon)$ versus $\beta$ with different $\varepsilon$ for the quadratic map, $z = 2$. The scaling behavior of $G(\beta, \varepsilon)$ when $\varepsilon \to 0$ can be found for $\beta \to -\infty$:

FIG. 4. The "free energy" $G(\beta, \varepsilon)$ vs $\beta$ for $z = 2$. The dotted line corresponds to $\beta_c = -1$.

$$G(\beta, \varepsilon) = -\frac{1}{n} \ln (l_i^{(n)})^\beta = \ln(2\varepsilon) + \varepsilon.$$  

(27)

For $\beta$ large enough and negative, we expect $l_i^{(n)}$ to still dominate the sum in Eq. (24). To investigate the scaling in $\varepsilon$ for $\beta < -1$, we have calculated $G(\beta, \varepsilon)$ as a function of $\varepsilon$ when $\varepsilon \to 0$. Figure 5 shows $G(\beta, 0) - G(\beta, \varepsilon)$ versus $\varepsilon$ in the case $z = 2$, for several values of $\beta < -1$. A linear scaling in $\varepsilon$ for $\beta < -1$ is found numerically:

$$G(\beta, 0) - G(\beta, \varepsilon) = M(\beta, \varepsilon),$$  

(28)

up to a neighborhood of $\beta_c = -1$. The slopes of the lines in Fig. 5 provide $M(\beta)$. The quantity $M(\beta) = [\partial G(\beta, \varepsilon) / \partial \varepsilon]_{\varepsilon=0}$ is analogous to a "zero field magnetization." For

FIG. 5. $G(\beta, 0) - G(\beta, \varepsilon)$ vs $\varepsilon$ in the case $z = 2$, for several values of $\beta < -1$. The numbers on each curve indicate the value of $\beta$. 

$$G(\beta, 0) - G(\beta, \varepsilon) = M(\beta, \varepsilon).$$  

(28)
\[ M(\beta) = \beta \] for \( \beta > -1 \), the function becomes singular.

The scaling in terms of \( D_q(\epsilon) \) can also be found in the

\[ D_q(\epsilon) = \frac{q}{2(\epsilon - 1)} - B_q \epsilon . \] (32)

This scaling has been confirmed numerically. Figure 8 shows the function \( B_q \) obtained in this way. For \( q < 2 \), the scaling changes approximately to \( 1 - D_q(\epsilon) \sim h(\epsilon) e^{1/2} \), close to \( \beta_e = 2 \), whereas for \( q \rightarrow -\infty \), we obtain

\[ D_{-\infty}(\epsilon) = \frac{n \ln 2}{\ln l^{(n)}_{\max}} \approx 1 - \frac{2}{\ln 2} \epsilon . \] (33)

**IV. CONCLUSIONS**

We have seen how different probability distributions on the same geometric support may lead to different multifractal properties and dissimilar scaling behaviors near the critical value of a parameter describing an attractor-repeller transition. The two Gibbs distributions that were studied correspond to different processes. A constant probability density, given by \( \sigma = 1 \), arises from a uniform distribution of initial conditions on the unit interval. The concept of escape rate enters naturally in the partition [Eq. (1)] and it determines the scaling of multifractal functions. In this case, we have shown that, re-
lated to the order of the map $z$, there exists a critical index $1/\gamma$ and a shift function $A_q(\varepsilon)$ associated with the change in multifractal properties at the transition. The scaling of the dimensions $D_q$ near the critical energy value $E_c$ for bifurcation to chaotic scattering has recently been investigated by Bieler, Grebogi, and Ott, who found that $1-D_q = a(q)/\ln \Delta$, where $\Delta = E_c - E$. This result and Eq. (18) suggest the existence of universality classes in the scaling of multifractal functions at the onset of a repeller.

A balanced measure, given by $\sigma = 0$, is obtained by iterating almost any initial condition on the repeller, and therefore it corresponds to the invariant probability distribution for that set. In the limit $\varepsilon \rightarrow 0$, the probability density presents a singularity at $l_{\text{min}}$, which is selected by the partition function above some critical value of the “temperature-like” parameters $\beta$ or $q$. This singularity determines the scaling behavior of $D_q(\varepsilon)$ above $q_c$, as given by Eq. (32). Thermodynamic quantities, such as $M(\beta)$ or $B_q$, then reflect the presence of a phase transition in $G(\beta,0)$ or $D_q(0)$ at $\beta_c$ or $q_c$. The knowledge of the scaling properties of multifractal functions near critical transitions may be relevant in their experimental measurement, where usually there are limitations in the location of critical values of parameters.

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