Onset of secondary flow in the modulated Taylor-Couette system

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The critical Reynolds number for the linear instability of primary flow is calculated for a Taylor-Couette system in which the rotation rate of either cylinder is modulated sinusoidally in time. The method used is based on that of Hall [J. Fluid Mech. **67**, 29 (1975)] and is restricted to small amplitudes of modulation but allows for a finite gap. For the case of outer-cylinder modulation, we find that the critical Reynolds number is larger than that for the unmodulated system, while, if the inner cylinder is modulated, it is smaller.

I. INTRODUCTION

There has been much effort, both experimental[1,2] and theoretical,[3–7] devoted to the problem of modulated Taylor-Couette flow. In 1987 Walsh and Donnelly performed a series of Taylor-Couette experiments with time periodic modulation[1,2] of the inner and outer cylinders. When the inner cylinder rotated at a constant angular velocity Ω_1, while the outer cylinder oscillated about zero mean rotation rate with velocity $\Omega_2=\epsilon\Omega_1\cos\omega t$, they found that basic flow was stabilized as compared to an experiment without modulation. This result qualitatively contradicted previous theoretical predictions.[4,5] In another experiment in which the angular velocity of the inner cylinder was modulated about nonzero mean while that of the outer cylinder was zero, they found that the basic flow was destabilized as compared to a steady rotation experiment. This result was in qualitative agreement with theory.[4,5]

In this paper, we perform linear stability analysis that employs a method based upon that of Hall.[3] We are restricted to small amplitude of modulation, but the narrow-gap approximation is not made. The qualitative results are that the primary flow becomes unstable at a Reynolds number higher than that for the unmodulated system, when the rotation rate of the outer cylinder is modulated about zero mean, whereas the critical Reynolds number is smaller than that for the unmodulated case when the rotation rate of the inner cylinder is modulated about nonzero mean.

In Sec. II, we outline the procedure for the calculation of the critical Taylor number for the case of modulation of the outer cylinder. No use is made of the narrow-gap approximation. We obtain a set of equations that are similar to those of Hall, and then we follow the procedure described by him[3] to calculate the shift of the critical Taylor number.

In Sec. III, we first compare the shift of the critical Reynolds number with experiment. Because the calculation for the case of inner-cylinder modulation is similar to that of outer-cylinder modulation, we directly present our result for inner-cylinder modulation case without giving the details of the calculation, and compare this with experiment. Finally, we also compare our results with the results of Carmi and Tustaniwskyj,[4,5] and the results obtained in the narrow-gap approximation.

II. CALCULATION OF THE SHIFT OF CRITICAL TAYLOR NUMBER (OUTER-CYLINDER MODULATION)

We begin with Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u} \tag{1a}$$

and continuity equation

$$\nabla\cdot\mathbf{u}=0 \ . \tag{1b}$$

In (1a) and (1b) $\mathbf{u}$ is the velocity field, ρ the mass density, p the pressure, and ν the kinematic viscosity. We use cylindrical-polar coordinates (r,θ,z) in our calculation, where the z axis is the axis of both outer and inner cylinders. The radius of the inner and outer cylinders is R_1 and R_2, respectively, and their ratio is defined to be η. Because the basic flow $\mathbf{V}$ is azimuthal, that is, it is of the form $(0,R_1\Omega_1V(r,t),0)$, $V(r,t)$ satisfies

$$\frac{\partial V}{\partial t}=\nu\left[\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}-\frac{V}{r^2}\right] , \tag{2a}$$

with the boundary conditions

$$V(R_1,t)=1 , \tag{2b}$$

$$V(R_2,t)=(\epsilon/\eta)\cos\omega t \ . \tag{2c}$$

Solving 2(a)–2(c) we obtain the basic velocity $V(r,t)$, which is

$$V(r,t)=V_s+\frac{\epsilon}{2}(V_1e^{i\omega t}+\tilde{V}_1e^{-i\omega t}) , \tag{3}$$

where $\tilde{V}_1$ is the complex conjugate of V_1, and V_s and V_1 are given by

$$V_s(r)=\frac{1}{1-\eta^2}\frac{R_1}{r}-\frac{\eta^2}{1-\eta^2}\frac{r}{R_1}\ , \tag{4a}$$

$$V_1=A_1I_1\left[\left[i\frac{\omega R_1^2}{\nu}\right]^{1/2}\frac{r}{R_1}\right]+A_2K_1\left[\left[i\frac{\omega R_1^2}{\nu}\right]^{1/2}\frac{r}{R_1}\right]. \tag{4b}$$

The I_1 and K_1 in (4a) and (4b) are modified Bessel functions and A_1 and A_2 are given by

$$A_1=-\frac{1}{\eta}K_1\left[\left[i\frac{\omega R_1^2}{\nu}\right]^{1/2}\right]\Big/\tilde{\Delta}\ , \tag{5a}$$

$$A_2=\frac{1}{\eta}I_1\left[\left[i\frac{\omega R_1^2}{\nu}\right]^{1/2}\right]\Big/\tilde{\Delta}\ , \tag{5b}$$

with $\tilde{\Delta}$ defined by

$$\tilde{\Delta}=I_1\left[\left[i\frac{\omega R_1^2}{\nu}\right]^{1/2}\right]K_1\left[\left[i\frac{\omega R_1^2}{\nu}\right]^{1/2}\Big/\eta\right]-I_1\left[\left[i\frac{\omega R_1^2}{\nu}\right]^{1/2}\Big/\eta\right]K_1\left[\left[i\frac{\omega R_1^2}{\nu}\right]^{1/2}\right]. \tag{6}$$

After obtaining the basic flow, we add in a perturbative velocity $\tilde{\mathbf{u}}$ and perturbative pressure $\tilde{p}$ into the Navier-Stokes equation and continuity equation. After neglecting the nonlinear terms, we obtain

$$\frac{\partial\tilde{\mathbf{u}}}{\partial t}+(\tilde{\mathbf{u}}\cdot\nabla)\mathbf{V}+(\mathbf{V}\cdot\nabla)\tilde{\mathbf{u}}=-\nabla\tilde{p}+\nu\nabla^2\tilde{\mathbf{u}} \tag{7a}$$

and

$$\nabla\cdot\tilde{\mathbf{u}}=0\ . \tag{7b}$$

Now we assume $\tilde{\mathbf{u}}=(u,(R_1^2\Omega/\nu)v,w)$ and u, v, w, and $\tilde{p}$ are θ independent. Furthermore, we define $\tilde{p}=\bar{p}\nu/R_1$, $z=R_1\bar{z}$, $r=R_1\bar{r}$, $\tau=\omega t$, and $\sigma=\omega R_1^2/\nu$ and put them into (7a) and (7b). We finally get

$$\sigma\frac{\partial u}{\partial\tau}-\frac{TV}{\bar{r}}v=-\frac{\partial\bar{p}}{\partial\bar{r}}+\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left[\bar{r}\frac{\partial u}{\partial\bar{r}}\right]-\frac{u}{\bar{r}^2}+\frac{\partial^2u}{\partial\bar{z}^2}\ , \tag{8a}$$

$$\sigma\frac{\partial v}{\partial\tau}+u\left[\frac{V}{\bar{r}}+\frac{\partial V}{\partial\bar{r}}\right]=\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left[\bar{r}\frac{\partial v}{\partial\bar{r}}\right]-\frac{v}{\bar{r}^2}+\frac{\partial^2v}{\partial\bar{z}^2}\ , \tag{8b}$$

$$\sigma\frac{\partial w}{\partial\tau}=-\frac{\partial\bar{p}}{\partial\bar{z}}+\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left[\bar{r}\frac{\partial w}{\partial\bar{r}}\right]+\frac{\partial^2w}{\partial\bar{z}^2}\ , \tag{8c}$$

$$\frac{\partial u}{\partial\bar{r}}+\frac{\partial w}{\partial\bar{z}}=0\ , \tag{8d}$$

where T is Taylor number defined by

$$T=\frac{2R_1^4\Omega_1^2}{\nu^2}\ . \tag{9}$$

We can also do the same kind of scaling to the basic flow, thus $\omega R_1^2/\nu$ in the (4)–(6) becomes σ, r/R_1 becomes $\bar{r}$, etc.

We assume the disturbance is periodic along z axis, that is,

$$u(\bar{r},\bar{z})=u(\bar{r})\cos a\bar{z}\ , \tag{10a}$$

$$v(\bar{r},\bar{z})=v(\bar{r})\cos a\bar{z}\ , \tag{10b}$$

$$w(\bar{r},\bar{z})=w(\bar{r})\sin a\bar{z}\ , \tag{10c}$$

$$\bar{p}(\bar{r},\bar{z})=\bar{p}(\bar{r})\cos a\bar{z}\ . \tag{10d}$$

By solving (8)–(10), we get

$$\left[DD_*-a^2-\sigma\frac{\partial}{\partial\tau}\right](DD_*-a^2)u=-a^2Tv\left[\frac{V_s(\bar{r})}{\bar{r}}+\frac{\epsilon e^{i\tau}}{2}\frac{V_1(\bar{r},\sigma)}{\bar{r}}+\text{c.c.}\right], \tag{11a}$$

$$\left[DD_*-a^2-\sigma\frac{\partial}{\partial\tau}\right]v=u\left[k_0+\frac{\epsilon\Phi(\bar{r},\sigma)e^{i\tau}}{2}+\text{c.c.}\right], \tag{11b}$$

$$u=\frac{\partial u}{\partial\bar{r}}=v=0,\quad \text{at } \bar{r}=1/\eta\ ,\ 1 \tag{11c}$$

where

$$D=\frac{\partial}{\partial\bar{r}}\ , \tag{12a}$$

$$D_*=\frac{\partial}{\partial\bar{r}}+\frac{1}{\bar{r}}\ , \tag{12b}$$

$$\Phi(\bar{r},\sigma)=-\left[\frac{V_1(\bar{r},\sigma)}{\bar{r}}+\frac{\partial V_1(\bar{r},\sigma)}{\partial\bar{r}}\right], \tag{12c}$$

$$k_0=\frac{2\eta^2}{1-\eta^2}\ . \tag{12d}$$

This formula is similar to (B1) in Appendix B of Ref. 3, so we follow the procedure of Ref. 3 to solve the problem.

The basic assumption is that when the system is at the point where the instability occurs, u and v should be periodic functions of time with frequency ω, so we can make Fourier expansion of them

$$u(\bar{r},\tau)=u_s(\bar{r})+\sum_{n=1}^{\infty}\tfrac{1}{2}[u_n(\bar{r})e^{in\tau}+\tilde{u}_n(\bar{r})e^{-in\tau}]\ , \tag{13a}$$

$$v(\bar{r},\tau)=v_s(\bar{r})+\sum_{n=1}^{\infty}\tfrac{1}{2}[v_n(\bar{r})e^{in\tau}+\tilde{v}_n(\bar{r})e^{-in\tau}]\ . \tag{13b}$$

Another assumption is that ϵ is small enough so that $u_s, v_s, T^c, u_1, v_1, \ldots$ can be expanded in terms of ϵ. In order to calculate the first nonzero correction of T^c, we can deal with u_s, v_s, T^c, u_1, and v_1 only.

Following Hall, we know

$$u_s(\epsilon,\bar{r})=u_s^0(\bar{r})+\epsilon^2 u_2^2(\bar{r})+\cdots \ , \tag{14a}$$

$$v_s(\epsilon,\bar{r})=v_s^0(\bar{r})+\epsilon^2 v_2^2(\bar{r})+\cdots \ , \tag{14b}$$

$$T^c(\epsilon)=T_0^c+\epsilon^2 T_2^c+\cdots \ , \tag{14c}$$

$$u_1(\epsilon,\bar{r})=\epsilon u_1^1(\bar{r})+\cdots \ , \tag{14d}$$

$$v_1(\epsilon,\bar{r})=\epsilon v_1^1(\bar{r})+\cdots \ . \tag{14e}$$

Also from the procedure similar to that of Hall, we know the equations that $u_s^0(\bar{r})$ and $v_s^0(\bar{r})$ satisfy are

$$M^2 u_s^0+\frac{a^2 T_c^0 V_s}{\bar{r}}v_s=0 \ , \tag{15a}$$

$$k_0 u_s^0 - M v_s^0 = 0 \ , \tag{15b}$$

$$u_s^0=\frac{\partial u_s^0}{\partial \bar{r}}=v_s^0=0, \quad \text{at } \bar{r}=1,1/\eta \tag{15c}$$

where M denotes

$$M=DD_* - a^2 \ . \tag{16}$$

The equations which $u_1^1(\bar{r})$ and $v_1^1(\bar{r})$ satisfy are

$$(M-i\sigma)Mu_1^1+\frac{a^2 T_c^0 V_s}{\bar{r}}v_1^1=-\frac{a^2 T_c^0 V_1}{\bar{r}}v_s^0 \ , \tag{17a}$$

$$(M-i\sigma)v_1^1 - k_0 u_1^1 = \Phi u_s^0 \ , \tag{17b}$$

$$u_1^1=\frac{\partial u_1^1}{\partial \bar{r}}=v_1^1=0, \quad \text{at } \bar{r}=1,1/\eta \ . \tag{17c}$$

The function pair $(\bar{u}_s^0,\bar{v}_s^0)$ adjoint to (u_s^0,v_s^0) satisfies the following differential system:

$$N^2 \bar{u}_s^0 + k_0 \bar{v}_s^0 = 0 \ , \tag{18a}$$

$$\frac{a^2 T_c^0 V_s}{\bar{r}}\bar{u}_s^0 - N\bar{v}_s^0 = 0 \ , \tag{18b}$$

$$\bar{u}_s^0=\frac{\partial \bar{u}_s^0}{\partial \bar{r}}=\bar{v}_s^0=0, \quad \text{at } \bar{r}=1,1/\eta \tag{18c}$$

where

$$N=\left[\frac{\partial}{\partial \bar{r}}-\frac{1}{\bar{r}}\right]\frac{\partial}{\partial \bar{r}}-a^2 \ . \tag{19}$$

Finally, by requiring the differential equations for u_s^2 and v_s^2 have solutions, we get T_2^c:

$$T_2^c=-\frac{1}{4a^2}\frac{\int_1^{1/\eta} a^2 T_c^0 \bar{u}_s^0[(V_1\bar{v}_1^1+\tilde{V}_1 v_1^1)/\bar{r}]d\bar{r}+\int_1^{1/\eta}\bar{v}_s^0(\Phi\bar{u}_1^1+\tilde{\Phi}u_1^1)d\bar{r}}{\int_1^{1/\eta}(\bar{u}_s^0 V_s v_s^0/\bar{r})d\bar{r}} \ . \tag{20}$$

III. RESULTS AND COMPARISON WITH EXPERIMENT

From the previous calculation, we know when ϵ is small, the critical Taylor number T^c can be written as

$$T^c(\epsilon,\sigma)=T_0^c+\epsilon^2 T_2^c(\sigma) \ , \tag{21}$$

and $T_2^c(\sigma)$ can be calculated as described above. Instead of using the critical Taylor number T^c and σ, Walsh and Donnelly use the critical Reynolds number $\mathcal{R}_1^c$ and γ to present their results.[1] The definition of γ is

$$\gamma=(d^2\omega/2\nu)^{1/2} \ , \tag{22}$$

and σ is defined as

$$\sigma=\frac{\omega R_1^2}{\nu} \ . \tag{23}$$

The relation between σ and γ is

$$\sigma=2\left[\frac{\eta}{1-\eta}\right]^2\gamma^2 \ . \tag{24}$$

The critical Reynolds number $\mathcal{R}_1^c$ is defined as

$$\mathcal{R}_1^c=(\Omega_1^c R_1 d/\nu)(d/R_1)^{1/2} \ , \tag{25}$$

where R_1 is the radius of the inner cylinder, d is the gap between two cylinders, and Ω_1^c is the critical angular velocity of the inner cylinder. Furthermore, another useful quantity is the stabilization Δ, which is defined by

$$\Delta(\epsilon,\gamma)=\frac{\mathcal{R}_1^c(\epsilon,\gamma)-\mathcal{R}_{10}^c}{\mathcal{R}_{10}^c} \ , \tag{26}$$

where $\mathcal{R}_1^c(\epsilon,\gamma)$ is the critical Reynolds number of the inner cylinder with modulation, and $\mathcal{R}_{10}^c$ is the critical Reynolds number of the inner cylinder without modulation. Because $\mathcal{R}_1^c \propto \Omega_1^c$ and $T_1^c \propto (\Omega_1^c)^2$, when $|\mathcal{R}_1^c(\epsilon,\gamma)-\mathcal{R}_{10}^c| \ll \mathcal{R}_1^c$,

$$\Delta(\epsilon,\gamma)=\frac{\epsilon^2}{2}\frac{T_2^c}{T_0^c} \ . \tag{27}$$

We have calculated the stabilization of systems with both outer and inner cylinder modulated. In each case we perform the required numerical calculation for two different radius ratios, namely, $\eta=0.88$ and $\eta=0.95$; ϵ is the same in all the calculations and equals 0.5. We also strictly follow the procedure in Appendix B of Ref. 3 to calculate the stabilization of both outer-cylinder modulated system and inner-cylinder modulated system in the case of a narrow gap. The ϵ in this calculation is also 0.5.

Results for the outer cylinder modulation case are presented in Fig. 1. From Fig. 1 we can see our result

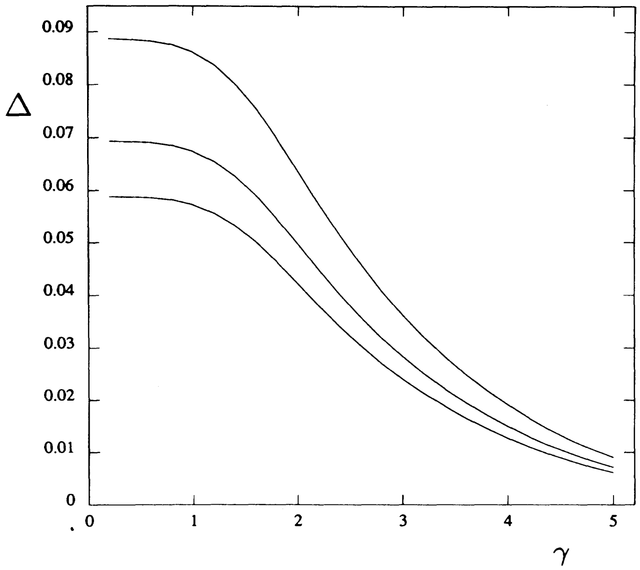

FIG. 1. Graphs of stabilization (Δ) vs γ for three outer-cylinder modulated systems with $\epsilon=0.5$. The top curve is for the system $\eta=0.88$, the central curve is for the system $\eta=0.95$, and the bottom one is for the system treated in the narrow-gap approximation.

qualitatively agrees with the result of narrow-gap approximation; and because the curve for the $\eta=0.95$ system lies closer to the curve of narrow-gap approximation than the curve for the $\eta=0.88$ system, we conclude that our result converges to the result of narrow-gap approximation when $\eta \rightarrow 1-$.

Results for the inner-cylinder modulation case are presented in Fig. 2. Because the calculational error of our method increases when γ approaches zero in the inner-cylinder modulation case, we have not used our method to calculate the stabilization for small γ; instead we follow an analytical procedure, which is also proposed by Hall,[3] to calculate the stabilization for $\gamma=0$, and put it on Fig. 2. From Fig. 2 we can see that our results for the inner-cylinder modulation case have the same qualitative behavior as that of the narrow-gap approximation for all γ, and the curve for the $\eta=0.95$ system lies closer to the curve of narrow-gap approximation than the curve for the $\eta=0.88$ system. Therefore we conclude that our result converges to the result of narrow-gap approximation as $\eta \rightarrow 1-$ for the inner-cylinder modulation case as well.

The comparison of our results of the outer-cylinder modulation case, with $\eta=0.88$ and $\epsilon=0.5$, with experiment[1,2] is presented in Fig. 3. From Fig. 3 we can see that both the experiment and the calculation give positive stabilization, and the difference between the experimental data and the calculational result is less than 0.02, which is 15% of the maximum value of the stabilization for $\gamma \leq 4$. Thus compared with the previous theories,[4,5] our calculation gives quite a good fit to the experiment.

For the inner-cylinder modulated system we do not present graphs to compare our results with the experiments because the difference between the two is too large. Although our results qualitatively agree with the experiments in that we obtain destabilization of the basic flow for $\gamma \leq 4$, the magnitude of the destabilization in the experiment[1,2] is almost 100 times as large as that in our theoretical calculation. We note, however, that there has been reported experimental[8] and theoretical[9] results with stabilization of the size predicted by Hall[3] and our result is of the same order as Hall's.

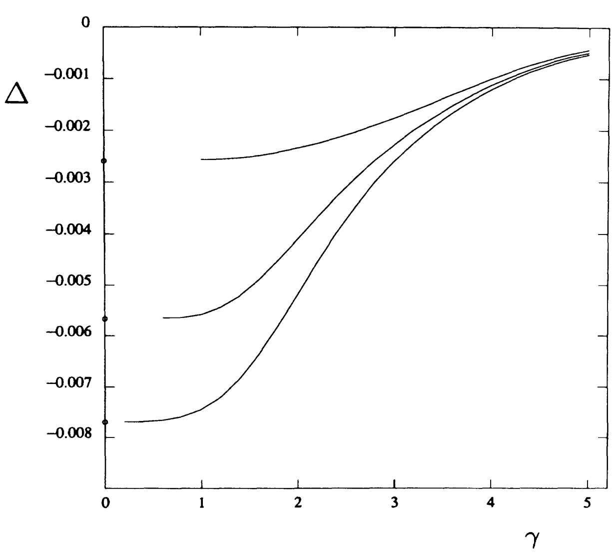

FIG. 2. Graphs of stabilization (Δ) vs γ for three inner-cylinder modulated systems with $\epsilon=0.5$. The top curve is for the system $\eta=0.88$, the central curve is for the system $\eta=0.95$, and the bottom one is for the system treated in the narrow-gap approximation. The points on the vertical axis denoted by circles were analytically calculated by the method of Ref. 3.

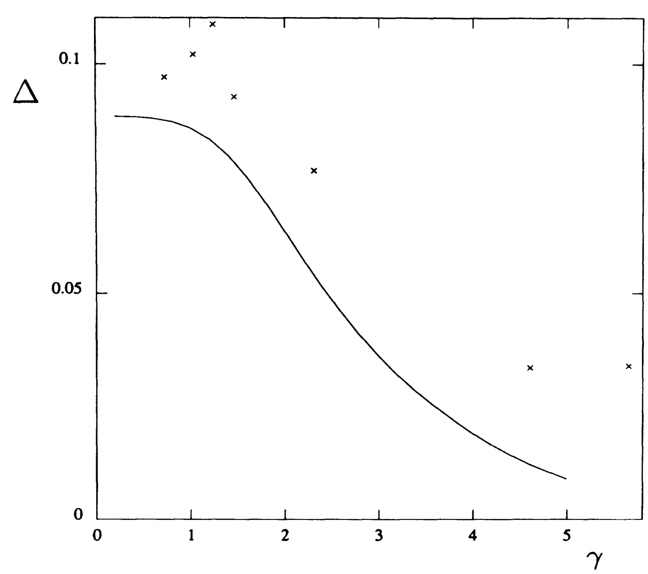

FIG. 3. Comparison of our calculation with Walsh and Donnelly's outer-cylinder modulated Taylor-Couette experiment. The parameters of the experiment are $\eta=0.88$ and $\epsilon=0.5$. The experimental points are denoted by $\times$.

Carmi and Tustaniwskyj[4,5] have calculated the stabilization in the case of both inner-cylinder modulation and outer-cylinder modulation. Their conclusion was that both inner- and outer-cylinder modulation would create destabilization for $\gamma \leq 5$. Their results contradict both the experiments and our results for the outer-cylinder modulation case. Both the experiment[1,2] and our calculation show that the outer-cylinder modulation creates stabilization for all γ.

The source of the discrepancy between the results of Ref. 4 and ours is not clear. We note that the present result of stabilization in the case of modulation of the outer cylinder is in accord with elementary arguments based on the stabilization curve[10] for the case of static rotations. See Refs. 1 and 2 for further discussion of this point.

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