Finite-size effects on the $f(\alpha)$ spectrum of the period-doubling attractor

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Quadratic maps are used to study the spectrum of singularities of the period-doubling attractor close to and at the onset of chaos. Finite approximations of the critical orbit have observable effects on the determination of $f(\alpha)$ which can be predicted by using the scaling properties of the attractor.

I. INTRODUCTION

Dissipative nonlinear systems under time evolution eventually settle into a subset of their phase space called an attractor. It is the attractor (or points on it) which is usually observed in an experiment. Recently, a formalism concerning the characterization of the entire attractor in phase space has been proposed. It consists of dividing a set embedded in a $d$-dimensional Euclidean space into $N$ disjoint pieces $S_i$, $i = 1, 2, \ldots, N$; such that each piece has a probability $p_i$, and it lies within a ball of radius $l_i$ in order to construct a partition function

$$\Gamma(q, \tau, |S_i|) = \sum_{i=1}^{N} \frac{p_i^{q}}{l_i^{q}}.$$  
(1)

It has been argued that when max $l_i \to 0$, $\Gamma$ tends to infinity for $\tau > \tau(q)$ and to zero for $\tau < \tau(q)$. The quantity $\tau(q)$ is related to the generalized dimensions $D_q$ considered in Ref. 2: $\tau(q) = D_q(q - 1)$. A convenient way of calculating $\tau(q)$ is to fix $\Gamma$ to a number as the partition is refined, and it is generally taken as $\lim_{N \to \infty} \Gamma(q, \tau, |S_i|) = 1$. Another function $f(\alpha)$, called the spectrum of singularities, and containing the same information as $D_q$ is also introduced. $f(\alpha)$ represents the fractal dimension of the subset of the attractor for which the probability measure at length $l$ scales as $p(l) \sim l^\alpha$, in the limit $l \to 0$. $\tau(q)$ can be converted into $f(\alpha)$ via a Legendre transformation,

$$\alpha = \frac{d}{dq} \tau(q), \quad f = q \alpha - \tau(q).$$  
(2)

It follows that $f(\alpha) \leq D_0$, the fractal dimension of the set. $D_q$ and $f(\alpha)$ are expected to be universal within scenarios associated with specific routes to chaos.

Measurements of $f(\alpha)$ have been reported in a forced Rayleigh-Bénard system$^{3,4}$ and in driven diode resonators$^2$ at the onset of chaos via both the period-doubling route and the quasiperiodic route with winding number equal to the golden mean. In an experimental situation, the location of the critical value of a parameter at the onset of chaos involves some uncertainty, and there is an additional limitation related to the number of data points defining the critical orbit. The effects that small deviations from criticality have on the observable spectrum $f(\alpha)$ for the quasiperiodic transition have been studied by Arneodo and Holschneider. The knowledge of these kind of effects is important since $f(\alpha)$ can be used to characterize a particular transition to chaos. Deviations from the expected $f(\alpha)$ curve which are not predicted by a model could suggest a new type of transition.

In this paper we consider the period-doubling transition to chaos and examine how $D_q$ and, consequently, $f(\alpha)$ are affected by the imprecision in the knowledge of the critical value of the bifurcation parameter and by the limited number of data points. In Sec. II the dimensions $D_q$ are analyzed as functions of the bifurcation parameter, for those values close to the accumulation point. The effect that a finite approximation of the critical orbit has on the $f(\alpha)$ curve is discussed in Sec. III and we make a comparison with the experimental measurements of Ref. 4. A universal function $A_q$ characterizing the deviation of any dimension $D_q$ from its theoretical asymptotic value is calculated.

II. PARAMETER DEVIATION

Dynamical systems that pass through a sequence of period doublings on their way to chaos can be represented by a family of one-parameter unimodal one-dimensional maps $x_{i+1} = f_i(x_i)$. A superstable orbit is generated by iterating the critical point $x_0$ of the map, given by $f_i'(x_0) = 0$. At values $\lambda = \lambda_n$, the set of iterates $x_i(\lambda_n) = f_i^{(n)}(x_0)$, $i = 1, \ldots, 2^n$, defines a $2^n$ cycle. The sequence $|\lambda_n|$ converges to $\lambda = \lambda_\infty$ according to $\lambda_n - \lambda_\infty \sim c \delta^{-n}$, where $\delta$ is a universal factor. The period-doubling attractor is the set of successive iterations of the map at $\lambda_n$. One can systematically build up this attractor by introducing levels with index $n$, corresponding to the $\lambda_n$'s, $n = 1, 2, 3, \ldots$, as in Fig. 1. On each level there are $2^n$ points $x_i(\lambda_n)$ making the orbit and $2^{n-1}$ intervals $I_n$ between adjacent points of the form

$$I_n(\lambda_n) = |x_i(\lambda_n) - x_{i+2^n} - (\lambda_n)|.$$  
(3)

The probabilities $p_i$ of these intervals are all equal. For $\lambda_n$ close to $\lambda_\infty$, the largest and the smallest intervals are $I_{\text{max}}(\lambda_n) = I_{2^n-1}(\lambda_n) - \alpha_{pd}^{-n-1}$ and $I_{\text{min}}(\lambda_n) = I_1(\lambda_n) - \alpha_{pd}^{2(n-1)}$ for quadratic maps, respectively, where

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\[ D_q(\lambda_n) = 2^{-n-1}(\lambda_n - \lambda_{n-1})^{\lambda_n} + \cdots, \]

do so that
\[ \frac{dD_q(\lambda_n)}{d\lambda_n} \approx \frac{1}{c} \delta^n[D_q(\lambda_n) - D_q(\lambda_{n-1})]. \]

An estimation of the difference factor can be obtained by taking the mean length \( \langle l(\lambda_n) \rangle = 2^{-n-1}\sum_{i=1}^{n-1} l_i(\lambda_n) \), as the average \( l_i \). We approximate it by using the two extreme scales as
\[ \langle l(\lambda_n) \rangle \sim 2^{-(n-1)(\alpha_{pd}) + \alpha_{pd}^2} n^{-1}. \]

With this assumption, Eq. (4) gives
\[ D_q(\lambda_n) \approx \frac{(n-1)\ln2}{\ln\langle l(\lambda_n) \rangle}, \]

and then
\[ \frac{dD_q(\lambda_n)}{d\lambda_n} \sim \frac{\delta^n}{(n-1)} \left[ \frac{\ln2}{\ln[\alpha_{pd} + \alpha_{pd}^2]/2] \right]^2. \]

In general, the factor in large parentheses will depend on \( q \). It can be seen from Eqs. (5) and (6) that under a variation of the bifurcation parameter, the greatest deviation from a critical value of a generalized dimension occurs for \( D_{\pm\infty}(\alpha_{max}) \). This corresponds to the rightmost end of \( f(\alpha) \). The smallest change occurs for \( D_{\pm\infty}(\alpha_{min}) \), or the left endpoint of \( f(\alpha) \).

### III. Variation of Number of Points on the Critical Orbit

For an experimental attractor obtained at the accumulation value of a period-doubling cascade, there is a limitation associated with the number of measured points taken to describe the \( 2^n \)-period orbit. In order to investigate this effect on \( D_q \) and on \( f(\alpha) \), consider the critical orbit \( x_i(\lambda_n) \), where \( i \) increases from 1 to \( 2^n \). The index \( m = 1, 2, \ldots \) defines a level of approximation of the period-doubling attractor, consisting of \( 2^m \) points and \( 2^{m-1} \) intervals of the form
\[ I_i^{(m)} = [x_i(\lambda_n) - x_{i+2^m}(\lambda_n)]. \]

This is illustrated in Fig. 1 for the value \( \lambda_{4n} \) with \( m \) varying from 1 to 4. To each level \( m \) there is associated a length scale of mean value \( \langle I_i^{(m)} \rangle \sim [(\alpha_{pd} + \alpha_{pd}^2)/2]^{2^m} - I_i(\lambda_n) \).

The partition function at level \( m \) is
\[ \Gamma_\lambda(q, r, \lambda_{in}) = 2^{-n-1} \sum_{i=1}^{2^m-1} (I_i^{(m)})^{-r} = 1. \]

We can consider Eq. (9) for fixed \( q \), and calculate \( D_q(\lambda_{in}) \) as a function of \( m \), for large \( m \). For \( q = \pm\infty \) we get
\[ D_{\infty}^{(m)} = -\frac{(m-1)\ln2}{\ln\langle I_i^{(m)} \rangle} \]
\[ = \frac{(m-1)\ln2}{(m-1)\ln\alpha_{pd} - \ln I_i(\lambda_{in})}, \]

and
\[ D_{\infty}^{(m)} = -\frac{(m-1)\ln2}{\ln\langle I_i^{(m)} \rangle} \]
\[ = \frac{(m-1)\ln2}{2(m-1)\ln\alpha_{pd} - \ln I_i(\lambda_{in})}. \]

\( D^{(m)} \), as functions of \( m \) will simultaneously increase or decrease depending on \( I_i(\lambda_{in}) = |x_i(\lambda_{in}) - x_{i+2^m}(\lambda_{in})| \), as they converge to the theoretical values

\[ D^{(m)}_{-\infty} = \frac{\ln2}{\ln\alpha_{pd}} = 0.75551 \ldots, \]

\[ D^{(m)}_{+\infty} = \frac{\ln2}{2\ln\alpha_{pd}} = 0.37775 \ldots. \]

Therefore the normalization of the map will affect the
way in which $D_q$ approaches its asymptotic value. For example, the universal function $g(x)$ satisfying Feigenbaum’s functional equation $g(x) = -\alpha_{pd} g(g(x/\theta_{pd}))$ (Ref. 8) is normalized with convention $g(0) = 1$, yielding $l_1(\lambda_\infty) = |g(0) - g^{(2)}(0)| = |1 - \alpha_{pd}^{2}| < 1$. Figure 2 shows $f(\alpha)$ calculated from the map $f_\lambda = \lambda(1 - 2x^2)$ at $\lambda_\infty = 0.837 005 134 \ldots$ (Ref. 8) with different numbers of points on the orbit. In this case, $l_1(\lambda_\infty) = 2\lambda_\infty^3 > 1$ and $(dD_q^{(m)}/dm) < 0$. The decreasing number of points produces an asymmetrical shift of the spectrum towards the right (Fig. 2). For $q = \pm \infty$, one obtains for large $m$

$$D_{-\infty} - D_{-\infty}^{(m)} \approx -\ln l_1(\lambda_\infty) \frac{\ln 2}{(\ln \alpha_{pd})^2} \frac{1}{(m - 1)} ,$$

$$D_{+\infty} - D_{+\infty}^{(m)} \approx -\ln l_1(\lambda_\infty) \frac{\ln 2}{4(\ln \alpha_{pd})^2} \frac{1}{(m - 1)} .$$

(13)

(14)

These relations suggest the following scaling for the deviation of any dimension $D_q$, at large $m$, from its asymptotic theoretical value

$$D_q - D_q^{(m)} \approx -\ln l_1(\lambda_\infty) \frac{1}{(m - 1)} A_q ,$$

(15)

where $A_q$ is a function characterizing the shift due to a finite approximation of each $D_q$. In particular,

$$A_{-\infty} = \frac{\ln 2}{(\ln \alpha_{pd})^2} = 0.823 48 \ldots ,$$

(16)

$A_{+\infty} = \frac{\ln 2}{4(\ln \alpha_{pd})^2} = 0.205 87 \ldots .$

Figure 3 shows $D_q^{(m)}(\lambda_\infty)$ versus $(m - 1)^{-1}$ for different values of $q$. For large $m$ (close to the $D_q$ axis), the curves approach straight lines in agreement with Eq. (15). We have linearized the curves in this region and investigated the $q$ dependence of the slopes, after dividing them by $-\ln l_1(\lambda_\infty)$. As shown in Fig. 4, the curve $A_q$ versus $q$ so obtained presents two asymptotes in the limit $q \rightarrow \pm \infty$.

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**FIG. 2.** $f(\alpha)$ at $\lambda_\infty$ from the map $x' = \lambda(1 - 2x^2)$ with different approximations of the critical orbit. The numbers indicate the value of $m$, with $2^n$ points being considered. The curves correspond to $m = 4$, $m = 6$, $m = 8$, $m = 14$, successively.

**FIG. 3.** The dimensions $D_q$ calculated at $\lambda_\infty$ with different numbers of points $2^n$ on the critical orbit, as functions of the inverse of $(m - 1)$. The number on each curve indicates the value of $q$. The same map as in Fig. 2 was used in the computations.

$$A_{-\infty} = \frac{\ln 2}{(\ln \alpha_{pd})^2} = 0.823 48 \ldots ,$$

$A_{+\infty} = \frac{\ln 2}{4(\ln \alpha_{pd})^2} = 0.205 87 \ldots .$

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**FIG. 4.** The shift function $A_q$ (which is proportional to the slopes of the curves in Fig. 3) vs $q$. 

which are likely to be $A_{+\infty}$ as computed before. The ratio $A_{-q}/A_{+q}$ ($q > 0$) is larger than 1 and tends asymptotically to the limit $A_{-\infty}/A_{+\infty} = 4$, which is the ratio of shifts of $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$, i.e., the endpoints of $f(\alpha)$. This implies that the dimensions $D_q$ with $q < 0$ are much more sensitive to a deviation from theoretical values than those with $q > 0$. The deviation increases as $q \to -\infty$, corresponding to the most rarefied part of the orbit. We notice that $f(\alpha)$ curves obtained from experimental period-doubling attractors also show this characteristic behavior.\textsuperscript{4,5} In Fig. 5 we present the experimental measurements of Glazier et al.\textsuperscript{4} and compare with our numerical approximation of the theoretical spectrum. The best fit corresponds to $f(\alpha)$ calculated at level $m = 4$; its associated mean length provides an indication of the length scales observed on the attractor. In Ref. 4, the error in the determination of $D_{+\infty}$ is $\sim 8\%$ and that of $D_{-\infty}$ is $\sim 20\%$, which gives $A_{-\infty}/A_{+\infty} \sim 20 D_{-\infty}/D_{+\infty} = 5$. Thus finite approximation of the orbit at the onset of chaos may account for the deviations from the theoretical $f(\alpha)$ found by using experimental data. The curve in Fig. 4 quantifies such an effect as well as the shifting of the $f(\alpha)$ spectrum observed in Fig. 2. A calculation using the map $f_\lambda = \lambda x (1 - x)$, for which $l_1(\lambda_{\infty}) < 1$, yields the same $A_q$ versus $q$ curve, indicating that the shift $A_q$ is a universal scaling property of the period-doubling attractor.

IV. CONCLUSIONS

We have found that both a deviation from the accumulation value of the bifurcation parameter and a finite approximation of the critical orbit have observable effects on the determination of global properties of the period-doubling attractor. By means of numerical and analytical considerations based on the scaling properties of the system, we have shown how one can account for deviations from the theoretical values in experimental measurements of the functions $f(\alpha)$ and $D_q$.

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