PHYSICAL REVIEW LETTERS

VOLUME 60 | 2 MAY 1988 | NUMBER 18

Information Transport in Spatiotemporal Systems

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(Received 3 November 1987)

Spatiotemporal chaos can be produced by complicated local dynamics in a small spatial region and observed globally through a process we call *information transport*. Information transport can be detected by computation of an information-theoretic quantity, the time-delayed mutual information, between measurements of the system at separate spatial points.

PACS numbers 05.45.+b, 02.60.+y, 03.40.−t, 06.50.−x

Finite-dimensional chaotic behavior[1] has been observed in several recent experiments on weakly turbulent fluid flows,[2] and it has been shown that bounded dissipative continuum systems such as viscous fluid flows and reaction-diffusion systems have finite-dimensional attractors.[3,4] The chaotic attractors seen in experiments have been obtained from data taken at a single point in a system; almost any spatial point yielded the same behavior. To understand how the observed chaotic behavior arises, however, it is important to sample continuum systems at several points simultaneously and to develop diagnostic tools that can reveal spatial variations in the dynamics.

Repeated measurements of finite accuracy on a chaotic system will never allow accurate prediction of the system state arbitrarily far into the future; an unresolvably small uncertainty in phase-space location will grow at an exponential rate and result in a complete loss of predictability of the system state. From an information-theoretic viewpoint, the system is a continuous source of new information, which is created at a rate equal to the metric entropy h_μ of the system.[5] In systems with only a few degrees of freedom, the exponential stretching mechanism can almost always be associated with well-defined directions in phase space. In spatiotemporal systems with infinite-dimensional phase spaces, it is of interest to determine the extent to which that mechanism can be spatially localized.

In this Letter we consider situations in which complicated dynamics localized in a small part of a system cause chaos everywhere—we call the process *information transport*. This should be a useful concept in systems where a local inhomogeneity or a boundary condition causes the entire system to behave chaotically. For example, a hot spot on the heated plate of a Rayleigh-Bénard cell or spatially dependent reaction terms in a reaction-diffusion system can result in chaotic behavior at all points in the system. We show that the time-delayed mutual information, an information-theoretic quantity that can be estimated from experimental time-series data, can determine whether the chaos in a spatiotemporal system is caused by spatially uniform effects or by information transport.

We begin by briefly reviewing the necessary information theory.[5-7] Suppose a dynamical system X has relaxed to an attractor which has an invariant probability distribution μ. A measurement of X induces a partition of the phase space of X and locates X in an element of the partition. The probability that an isolated measurement will find the system in the ith element of the partition is the integral of μ over that element, $p(i)$. If two systems X_1 and X_2 are measured simultaneously, then the relevant probability distributions are $p(i_1)$, $p(i_2)$, and the joint distribution $p(i_1,i_2)$. These distributions yield a dynamical invariant,

$$I(X_1;X_2)=\sum_{i_1,i_2} p(i_1,i_2)\log_2\left[\frac{p(i_1,i_2)}{p(i_1)p(i_2)}\right], \tag{1}$$

the *mutual information*, which is the amount of information gained about one system from a measurement of the other; the sum in (1) extends over all elements of the joint partition for which $p(i_1)$ and $p(i_2)$ are both nonzero. The mutual information is given in units of bits of information. If X_1 and X_2 are independent, then $p(i_1,i_2)=p(i_1)p(i_2)$ and $I(X_i;X_2)=0$.

The higher-dimensional analog of mutual information is the *redundancy*,

$$R_N(X_1,\ldots,X_N) = \sum_{i_1,\ldots,i_N} p(i_1,\ldots,i_N)\log_2\left[\frac{p(i_1,\ldots,i_N)}{p(i_1)\cdots p(i_N)}\right]. \quad (2)$$

If system $\mathbf{X}_1$ is m_1 dimensional and $\mathbf{X}_2$ is m_2 dimensional, then the mutual information between measurements of the two systems is

$$I(\mathbf{X}_i;\mathbf{X}_2)=R_{m_1+m_2}(\mathbf{X}_1,\mathbf{X}_2) - R_{m_1}(\mathbf{X}_1)-R_{m_2}(\mathbf{X}_2). \quad (3)$$

For the numerical tests in this Letter we use an algorithm developed by Fraser[6,7] to estimate the mutual information and redundancy of sets of experimental measurements.

Suppose measurements with finite accuracy are made at distinct spatial points A and B in a chaotic spatiotemporal system and the system attractor is reconstructed from the data at each site.[8,9] Suppose further that the chaos in the system is due to some local effect at A. The information brought up from infinitesimal scales in the field variables in a short time interval before time t will in general be contained in larger-amplitude scales at A than at B and time t. Finite-accuracy measurements at time t will not resolve at B all of the new information which is observed by the measurements at A. For example, in a reaction-diffusion system the value of a field variable at A at time t is transmitted infinitely quickly to any other spatial point B by diffusion, but with an infinitesimal amplitude. If localized dynamics at A results in information generation, then with finite-accuracy measurements it will take some time for the information which is observable at A at time t to be observable at B; that time is, roughly, the diffusive time L^2/D, where L is the spatial separation of A and B, and D is a diffusion coefficient. Thus a reasonable physical explanation of the mechanisms that make such a system chaotic is one of localized information production and subsequent spatial transport.[10]

Information transport can be detected by computation of the time-delayed mutual information $I(\mathbf{A}_t;\mathbf{B}_{t+T})$ between measurements at two sites at different times.[11] The amplitude scales that contain the information observed at A at time t will become resolvable at B at some later time $t+T$; so the joint probability distribution between measurements at A at t and measurements at B at $t+T$ will become more sharply peaked. Thus, $I(\mathbf{A}_t;\mathbf{B}_{t+T})$ will be larger than $I(\mathbf{A}_t;\mathbf{B}_t)$. The graph of $I(\mathbf{A}_t;\mathbf{B}_{t+T})$ will not be symmetric about $T=0$,[12] but will attain a maximum value for some $T_{\max}>0$. On the other hand, if no localized effects exist, the graph of $I(\mathbf{A}_t;\mathbf{B}_{t+T})$ will be symmetric about $T=0$.

We now test the information-transport concept on a model one-dimensional reaction-diffusion system:

$$\partial_t\mathbf{u}(z,t)=D\,\partial_{zz}\mathbf{u}(z,t)+\mathbf{F}(z,\mathbf{u}) \quad (4)$$

with periodic boundary conditions at $z=\pm L/2$, where $\mathbf{u}(z,t)$ is the vector of species concentrations, D is the diffusion coefficient (the same for all species), and $\mathbf{F}(z,\mathbf{u})$ is the local reaction term, for which we use the three-variable Rössler model[13]

$$\begin{aligned} F_1(z,\mathbf{u})&=-u_2(z,t)-u_3(z,t),\\ F_2(z,\mathbf{u})&=u_1(z,t)+au_2(z,t),\\ F_3(z,\mathbf{u})&=b+u_3(z,t)[u_1(z,t)-c(z)], \end{aligned} \quad (5)$$

where $a=0.15$, $b=0.20$, and the parameter c varies spatially as follows:

$$c(z)=\begin{cases}5.8, & |z|>0.02L,\\ 18.0, & |z|\le 0.02L.\end{cases} \quad (6)$$

For $c=5.8$ the uncoupled Rössler model has a limit-cycle attractor, while for $c=18.0$ the attractor is chaotic, with $h_\mu=0.025$ bit/s. The mean orbital period for both attractors is approximately 6.1 s. In (4) we set $L=1.0$ cm and $D=1.0\times10^{-5}$ cm^2/s. The spatial extent of the system was modeled by a grid of 500 points, with use of centered differencing to compute the spatial derivative. An Adams-Bashforth-Moulton variable-order variable-stepsize integration technique[14] was used to evolve the system with an accuracy of at least six significant figures. Trials using finer grids and/or a Gear method of integration yielded the same behavior.

The temporal evolution of the system after transients have decayed is shown in Fig. 1(a). The large-scale motion of the system consists of wave trains originating at $z=\pm L/2$ and traveling to the center from both sides at an average wave speed of 1.79×10^{-2} cm/s. A region of large-amplitude oscillations in the central region absorbs the wave trains. The system is chaotic at all spatial points: The large-amplitude oscillations in the central region essentially reproduce the behavior of the uncoupled chaotic Rössler model, and there is a chaotic variation in peak heights on the wave trains. The amplitude of the chaotic variation at $z=\pm L/2$ is roughly 60 times smaller than near $z=0$; this variation is shown by next-maximum return maps at two spatial sites in Figs. 1(b) and 1(c). A partial Lyapunov exponent spectrum for (4) was computed: $\{\lambda_i\}=\{0.077, 0.000, -0.010, -0.024, -0.035, -0.036, \ldots\}$ in units of bits per second; so $h_\mu=0.077$ bit/s, and the Lyapunov dimension[15] of the system attractor is 5.2.

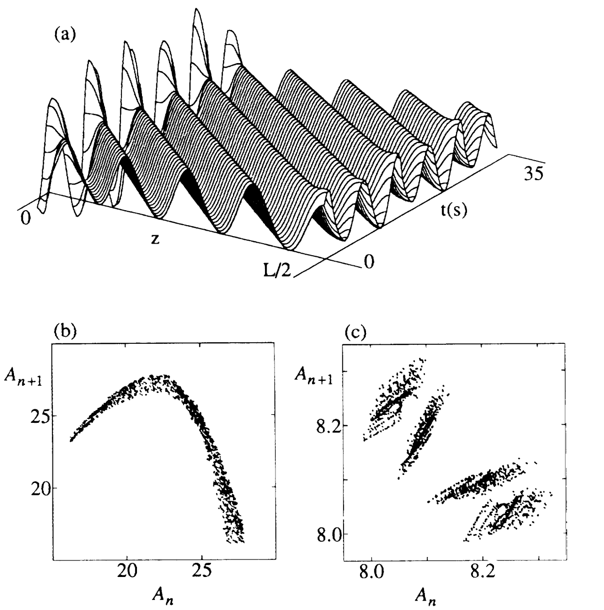

FIG. 1. (a) The evolution of the concentration $u_1(z,t)$ of the model system (4). The system is symmetric about $z=0$. Return maps for the amplitudes $A_n = [u_1^{\max}(z_0,t)]_n$ are shown for (b) $z_0=0$ and (c) $z_0=0.4L$.

The system attractor was reconstructed from 65536-point time series of the u_1 coordinate at each spatial point. The sampling period was 0.1 s and the data were truncated to six significant figures. Fraser's criteria[6,7] were used to find the optimal time delay and embedding dimension for time-delay reconstruction and to estimate h_μ at each site. The optimal time delay varied in the range 1.3 to 1.5 s. The metric entropy was the same (to within the accuracy of the estimation technique) at all spatial sits: $h_\mu \approx 0.085$ bit/s. Thus from measurements at any point in the system we can recover the metric entropy of the whole system to within 10%, which is good agreement given the data requirements discussed in Ref. 7.

Figure 2 shows the estimated time-delayed mutual information[16] between three-dimensional reconstruction[17] at the site $z=0$ and two other sites. There is a definite asymmetry with respect to $T=0$; the peak at positive T indicates that information is transported *outward* from the center. Figure 3(a) shows $T_{\max}$ as a function of distance from $z=0$. Information created in the center of the system is transported by a traveling wave in a direction opposite to the large-scale waves in the system. The effective information wave speed is 9.2×10^{-4} cm/s, which is 20 times slower than the speed of the large-scale waves propagating in the opposite direction. The full width at half maximum of the mutual-information peak is shown to vary as the square root of distance in Fig. 3(b). Since the transport velocity is constant, the peak width grows as the square root of elapsed time. Thus information diffuses as it is transported by the traveling

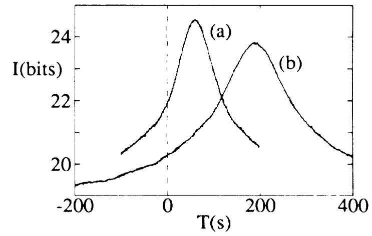

FIG. 2. Graphs of the estimated (Ref. 16) time-delayed mutual information $I(\mathbf{A}_t, \mathbf{B}_{t+T})$ for site A at $z=0$ and site B at (a) $z=0.06$ and (b) $z=0.18$. The peaks occur at positive T, indicating that information is transported from site A to site B.

wave; the effective diffusion coefficient is 7.7×10^{-5} cm^2/s. All of these results were robust to changes of reconstruction delay time and increases of the embedding dimension at either site, resolution of the sampled variable at either site, and number of data points used.

The utility of the information-transport idea is illustrated in Fig. 4, which depicts the temporal evolution of an initial perturbation to the $z=0$ site. Once the perturbation has organized in the central region, it spreads outward at a constant rate of 9.5×10^{-4} cm/s. Thus the information-transport rate we find from time-delayed mutual-information estimates agrees with the physically observable rate at which a perturbation spreads in the system.

Correlation functions are another tool for the observation of relationships between variables. However, computations of nth-order correlation functions (for $n \le 4$) between the reconstructed attractors have failed to reveal

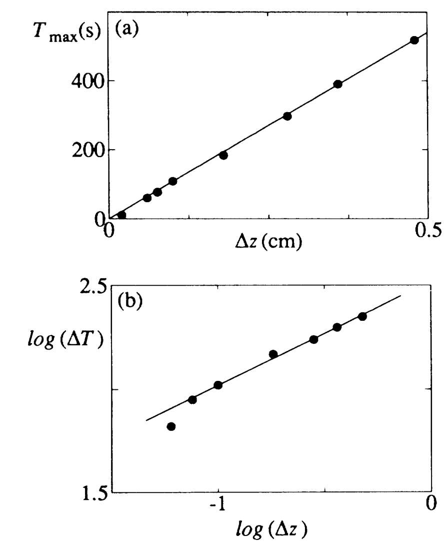

FIG. 3. (a) The delay time $T_{\max}$, which corresponds to the maximum in the estimated time-delayed mutual information, as a function of Δz (the distance from $z=0$). (b) The full width at half maximum (ΔT) of the peak as a function of Δz. The slope of the fitted curve is 0.497.

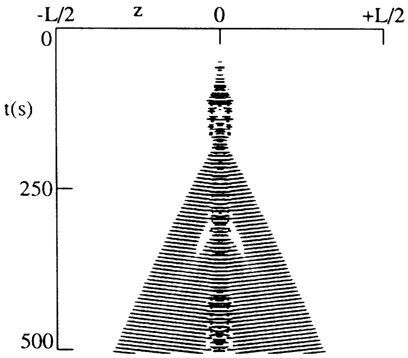

FIG. 4. The evolution of a small (1%) δ-function perturbation applied to the system at $z=0$. A site is blackened if the difference between the original solution and the perturbed solution is greater than an arbitrary cutoff. Notice that waves continually form at the farthest distance from the center that the perturbation has reached and then propagate *towards* the center.

any of the information-transport effects observed in the mutual-information graphs. Thus mutual information can find relationships that correlation functions cannot.[6,7]

We have shown that information transport can be observed in a chaotic system when the cause of the chaotic behavior is spatially localized. We studied a closed system in this Letter, but the idea of information transport can be applied to open systems if the probability distributions considered are stationary with respect to temporal translation. The concept of information transport that we have developed can be used to gain insight on the mechanisms that create chaos in spatiotemporal systems.

We gratefully acknowledge many useful discussions with Andrew Fraser. Numerical modeling was carried out on the Cray XMP at the University of Texas Center for High Performance Computing. This research was supported by a grant from the U.S. Department of Energy Office of Basic Energy Sciences. One of us (J.V.) acknowledges the support of an Exxon Fellowship.

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[12] The statistics on the attractors are stationary, so that $I(\mathbf{A}_{t+T};\mathbf{B}_t)=I(\mathbf{A}_t;\mathbf{B}_{t+T})$.

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[16] The time-delayed mutual information is obtained from the time-delayed redundancy [cf. (3)] by subtraction of a fixed offset equal to the redundancy of each individual reconstruction. The technique we use underestimates redundancy, and so our estimate of the mutual information is only accurate up to a fixed offset. However, what is important is not the magnitude but the variation of the mutual information with delay time.

[17] Calculations of the mutual information (Ref. 7) between *components* of the reconstructed vectors indicated that a three-dimensional reconstruction was sufficient at all spatial sites. This is obviously incorrect from a strictly theoretical viewpoint, since the system attractor is 5.2-dimensional; what the calculations indicate is that the full dimensionality of the attractor can only be seen on smaller amplitude scales than those resolved by the six-significant-figure data we used. Reconstructions in three dimensions yielded accurate estimates of h_μ (from mutual-information calculations) and of the largest Lyapunov exponent λ_1 [by the method of A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Physica (Amsterdam) **16D**, 285 (1985)]. We therefore used three-dimensional reconstructions for most of the computations; checks using six-dimensional reconstructions revealed no differences whatsoever.