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THE ENERGY OF A STEADY-STATE CRACK IN A STRIP

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ABSTRACT

WE CALCULATE the total energy of a semi-infinite crack moving in a two-dimensional brittle-elastic strip. First we prove a virial theorem relating kinetic and potential energy. Second, we use the Wiener–Hopf technique to find the stress fields surrounding the crack. The energy is computed exactly. Almost all of the computation can be accomplished without carrying out the Wiener–Hopf decomposition explicitly. We describe a numerical technique by which to perform the Wiener–Hopf decomposition when needed. Finally, an adiabatic argument allows one to deduce an equation of motion for the crack. We show that energy flux to the tip must depend upon acceleration in our geometry, and compute the dependence explicitly.

1. INTRODUCTION

THE FIRST calculation intended to predict the speed of a crack traveling in a brittle plate was performed by MOTT (1948). It rested upon an estimation of the total energy surrounding the crack. As the crack extends, it releases elastic potential energy in the material surrounding it. Much of the energy is used by the crack to create the new surfaces which define it. Simultaneously, the crack faces move apart, causing the material about them to deform, and costing both kinetic and potential energy. The balance of these various energies allows one to predict the speed of the crack. Mott was able to estimate that the speed of the crack should approach some velocity $v_{\max}$ comparable to the speed of sound as

$$v(t) = v_{\max}\sqrt{(1-l_0/l(t))}.$$

where l_0 is the length of the crack at $t = 0$ when it is stationary, and $l(t)$ is the length at subsequent times. Proceeding by dimensional analysis alone, he was not able to determine $v_{\max}$. Following attempts to improve on the approximations (ROBERTS and WELLS, 1954; DULANEY and BRACE, 1960), it was determined that the maximum velocity of a crack should be the Rayleigh wave speed. The physical reason for this limit was first emphasized by STROH (1957); crack motion can be viewed as the propagation of a wave upon a free surface, for which the Rayleigh wave speed has long been known to be the limiting velocity.

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In what follows, we will repeat Mott's calculation in a geometry somewhat different from the one he considered. We will first compare our method for finding the energy with one introduced by FREUND (1972b), and show that the conventional application of Freund's results to our geometry violates an elementary theorem of elasticity. In fact, Freund's results are not supposed to apply to our geometry. The calculation of the energy will then be performed without approximation by another method. We will finally discuss the implications of the energy calculation for the motion of the crack.

2. VIRIAL THEOREM

Consider a crack moving in steady state at a constant velocity v in an ideal brittle-elastic strip, as shown in Fig. 1. The upper and lower edges of the plate are displaced vertically upwards by a constant amount δ compared to their unstressed positions. To say that the strip is brittle-elastic means that all points in the strip obey the equations of linear elasticity until the instant bonds separate. We will begin by proving a simple virial theorem which simplifies energy calculations considerably. We denote by $u_x(x, y)$ and $u_y(x, y)$ the displacement of the point now at $(x+u_x, y+u_y)$ from its original unstressed location, and by $\sigma_{\alpha\beta}(x, y)$, where α and β take on values x and y, the stress at (x, y). The kinetic energy of a moving crack in a strip of thickness w and density ρ is

$$K = \tfrac{1}{2} w\rho \int_{-1}^{1} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x \sum_{\alpha} \dot{u}_{\alpha}^{2},$$

while the potential energy is

$$\begin{aligned} P &= \tfrac{1}{2} w \int_{-1}^{1} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x \sum_{\alpha\beta} \sigma_{\alpha\beta} \frac{\partial u_{\alpha}}{\partial \beta} \\ &= \tfrac{1}{2} w \int \mathrm{d}x\, \mathrm{d}y \sum_{\alpha\beta} \frac{\partial}{\partial \beta} [u_{\alpha}\sigma_{\alpha\beta}] - \tfrac{1}{2} w\rho \int \mathrm{d}x\, \mathrm{d}y \sum_{\alpha} \ddot{u}_{\alpha} u_{\alpha}, \end{aligned}$$

where we have used the equation of motion

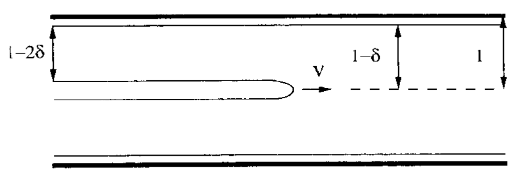

FIG. 1. A semi-infinite crack travels at velocity v in a strip of unit half-width. The top and bottom edges of the strip are fixed at height δ above their initial positions.

$$\rho \ddot{u}_\alpha = \sum_\beta \frac{\partial \sigma_{\alpha\beta}}{\partial \beta}.$$

It follows then that the total energy is given by

$$E = K+P = \tfrac{1}{2} w\rho \int \mathrm{d}x\,\mathrm{d}y \sum_\alpha [(\dot{u}_\alpha)^2 - u_\alpha \ddot{u}_\alpha] + \tfrac{1}{2} w \int_S \mathrm{d}\hat{n}\cdot\mathbf{e}, \tag{1}$$

with the last integral over the boundaries of the system, and

$$e_\beta = \sum_\alpha u_\alpha \sigma_{\alpha\beta}.$$

We use this result to simplify the problem of a crack moving in the steady state. The time derivatives can all be converted to spatial derivatives, and integrating by parts we have

$$E = v^2 \rho w \int \mathrm{d}x\,\mathrm{d}y \sum_\alpha \left(\frac{\partial u_\alpha}{\partial x}\right)^2 + \tfrac{1}{2} w \int_S \mathrm{d}\hat{n}\cdot\mathbf{e}. \tag{2}$$

We have used the fact that $\partial u_\alpha/\partial x$ decays to zero at $\pm\infty$. The surface integral accounts for the energy contained in the stressed plate off to the right, while the remaining energy contained in kinetic and potential fields surrounding the crack can be computed by finding the kinetic energy alone. By avoiding the need to compute potential energy separately, the labor required to find the energy of a crack is reduced by about a factor of three. A related conservation integral is given by FREUND (1972c), and several more are summarized by NILSSON and STAHLE (1988).

3. WIENER–HOPF SOLUTION

We now determine the stress fields surrounding a crack moving at constant velocity v by the Wiener–Hopf technique (NOBLE, 1959; KNAUSS, 1966). The calculation of the stresses surrounding the crack in this geometry is not new (FREUND, 1972a, b; NILSSON, 1972); we will repeat it briefly since we will need certain intermediate results to explain our work. We are interested in two related problems. In both cases, the symmetry of the problem allows one to restrict attention to the upper half plane alone. The first problem, which we shall refer to as problem (A) since all the stresses are after the crack, features a crack moving in steady state at velocity v in a strip obeying boundary conditions

$$u_y(x,1) = \delta \tag{3a}$$

$$u_x(x,1) = 0 \tag{3b}$$

$$\sigma_{xy}(x,0) = 0 \tag{3c}$$

$$\sigma_{yy}(x,0) = 0, \quad \text{for} \quad x<0 \tag{3d}$$

$$u_y(x,0) = 0, \quad \text{for} \quad x>0. \tag{3e}$$

The second problem (B) differs from the first only because one subtracts from stress and displacement fields the solution for a uniformly stressed plate with a crack ; now all stresses are behind the crack. Stresses in a uniform plate may be found from the constitutive equations

$$\sigma_{\alpha\beta} = \rho\left\{(c_l^2-2c_t^2)\sum_\gamma \delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial \gamma}+c_t^2\left(\frac{\partial u_\alpha}{\partial \beta}+\frac{\partial u_\beta}{\partial \alpha}\right)\right\}, \tag{4}$$

which are expressed in terms of the longitudinal and transverse sound speeds, c_l and c_t. We mean by c_l the longitudinal wave speed appropriate for a thin plate, given in terms of the bulk Young's modulus E, the Poisson ratio ν and the bulk density ρ as $c_l^2 = E/(\rho(1-\nu^2))$. One also has $c_t^2 = E/(2\rho(1+\nu))$. For a uniformly stressed plate

$$u_x = 0, \quad u_y = \delta y, \quad \sigma_{yy} = \rho\delta c_l^2 \equiv \sigma_\infty, \quad \sigma_{xx} = \rho\delta(c_l^2-2c_t^2), \quad \sigma_{xy} = 0. \tag{5}$$

Subtracting these fields is equivalent to imposing boundary conditions

$$u_y(x,1) = 0 \tag{3a$'$}$$

$$u_x(x,1) = 0 \tag{3b$'$}$$

$$\sigma_{xy}(x,0) = 0 \tag{3c$'$}$$

$$\sigma_{yy}(x,0) = -\sigma_\infty, \quad \text{for} \quad x<0 \tag{3d$'$}$$

$$u_y(x,0) = 0, \quad \text{for} \quad x>0. \tag{3e$'$}$$

It is most convenient to employ the Wiener–Hopf technique on problem (B); all of our equations for stress and strain will be for problem (B), and if we wish information of problem (A), we will add the required fields explicitly.

The equations of motion for an elastic medium may be expressed in terms of two scalar potentials which obey the wave equation. One of them is a potential for longitudinal waves, the other is the potential for transverse waves. From them one may derive the displacements, since

$$\mathbf{u} = \nabla v_l + \nabla \times v_t. \tag{6}$$

In the steady state one has

$$\alpha^2\frac{\partial^2 v_l}{\partial x^2}+\frac{\partial^2 v_l}{\partial y^2} = 0,$$

$$\beta^2\frac{\partial^2 v_t}{\partial x^2}+\frac{\partial^2 v_t}{\partial y^2} = 0, \tag{7}$$

where $\alpha^2 = 1-v^2/c_l^2$, and $\beta^2 = 1-v^2/c_t^2$. Then one can write

$$v_l = A_{sl}(k)\sinh \alpha k y + A_{cl}(k)\cosh \alpha k y,$$

$$v_t = A_{st}(k)\sinh \beta k y + A_{ct}(k)\cosh \beta k y. \tag{8}$$

In terms of these constants one has

$$u_x = A_{ct}(k)\beta k \sinh \beta k y + A_{st}(k)\beta k \cosh \beta k y \\ - ik(A_{sl}(k)\sinh \alpha k y + A_{cl}(k)\cosh \alpha k y), \quad (9a)$$

$$u_y = ik(A_{st}(k)\sinh \beta k y + A_{ct}(k)\cosh \beta k y) \\ + \alpha A_{cl}(k)k \sinh \alpha k y + \alpha A_{sl}(k)k \cosh \alpha k y. \quad (9b)$$

One can work out the stresses using (4) and (9).

Three of the coefficients $A_{cl}(k) \dots A_{st}(k)$ can be found from the three boundary conditions which apply to all x, and hence to all k (3a′–c′). Define

$$u_y(k,0) \equiv u_y^0(k). \quad (10)$$

Then

$$A_{st}(k) \equiv a_{st}(k)u_y^0(k) = \frac{i(2\alpha\beta s_\alpha s_\beta - 2c_\alpha c_\beta + \beta^2 + 1)u_y^0(k)}{B(k)} \quad (11a)$$

$$A_{cl} \equiv a_{cl}u_y^0(k) = -\frac{((\beta^2+1)s_\alpha s_\beta - \alpha\beta(1+\beta^2)c_\alpha c_\beta + 2\alpha\beta)u_y^0(k)}{\alpha B(k)} \quad (11b)$$

$$A_{ct} \equiv a_{ct}u_y^0(k) = \frac{2iu_y^0(k)}{(\beta^2-1)k} \quad (11c)$$

$$A_{sl} \equiv a_{sl}u_y^0(k) = \frac{(\beta^2+1)u_y^0(k)}{(\alpha\beta^2-\alpha)k}, \quad (11d)$$

with

$$B(k) = (\beta^2-1)k(c_\alpha s_\beta - \alpha\beta s_\alpha c_\beta). \quad (11e)$$

We have used the abbreviated notation

$$s_\alpha = \sinh \alpha k, \quad s_\beta = \sinh \beta k, \quad c_\alpha = \cosh \alpha k, \quad c_\beta = \cosh \beta k.$$

One now uses the Wiener–Hopf trick; define

$$\sigma_{yy}^+ = \int_0^\infty \sigma_{yy}(z)e^{ikz}\,\mathrm{d}z, \quad \sigma_{yy}^- = \int_{-\infty}^0 \sigma_{yy}(z)e^{ikz}\,\mathrm{d}z,$$

with u_y^+ and u_y^- defined similarly. One should note that σ_{yy}^- has no poles in the lower half plane, σ_{yy}^+ no poles in the upper half plane. Similarly u_y^+ has no poles in the upper half plane, and u_y^- has no poles in the lower half plane.

One writes that

$$\sigma_{yy}(k) = \sigma_{yy}^+ + \sigma_{yy}^- = \sigma_{yy}^+ - \frac{\sigma_\infty}{ik},$$

$$u_y(k) = u_y^-.$$

Then defining

$$F(k) = -\frac{\sigma_{yy}}{u_y}, \tag{12}$$

we have

$$-F(k)u_y^- = \sigma_{yy}^+ - \frac{\sigma_\infty}{ik}. \tag{13}$$

The point of defining $F(k)$ lies in the fact that, since it is a ratio of two quantities expressible in terms of the $A_{cl} \ldots A_{st}$, the unknown function $u_y^0(k)$ which appears in all of these does not matter. The function $F(k)$ is

$$F(k) = k\rho c_t^2 \frac{s_\alpha s_\beta\{(\beta^2+1)^2+4(\alpha\beta)^2\} - c_\alpha c_\beta \alpha\beta\{(\beta^2+1)^2+4\} + 4\alpha\beta(\beta^2+1)}{\alpha(1-\beta^2)(\alpha\beta s_\alpha c_\beta - c_\alpha s_\beta)}. \tag{14}$$

For small k

$$F \to f_0 = \rho c_l^2, \tag{15}$$

while for large real k

$$F \to |k| f_\infty = \frac{|k|\rho c_t^2}{\alpha(\beta^2-1)}\{(\beta^2+1)^2 - 4\alpha\beta\}. \tag{16}$$

Let us suppose that we can write $F(k)$ in the following way, as

$$F(k) = \frac{F^-(k)}{F^+(k)}, \tag{17}$$

where F^- has no poles in the lower half plane, and F^+ has no poles in the upper half plane. Then we can write

$$-kF^-(k)u_y^- = k\sigma_{yy}^+ F^+(k) + i\sigma_\infty F^+(k). \tag{18}$$

One has set equal an expression with no poles in the upper half plane to one with no poles in the lower half plane. Therefore, both must equal a constant. The constant can be fixed by examining the behavior of the expressions for $k \to 0$. Notice that

$$\lim_{k \to 0} u_y^- \sim \frac{\delta}{ik}; \tag{19}$$

this statement follows from the fact that u_y vanishes for large positive x and goes to δ for large negative x. So one has

$$u_y^0(k) = u_y^-(k,0) = u_y(k,0) = \frac{\delta F^-(0)}{ikF^-(k)}. \tag{20}$$

The problem is now solved, apart from the difficulties of decomposing F into F^+ and F^-.

The following equation is frequently useful. We have that

$$F(k) = F(-k) \Rightarrow \frac{F^-(k)}{F^+(k)} = \frac{F^-(-k)}{F^+(-k)} \Rightarrow F^-(k)F^+(-k) = \text{const.}$$

The constant is arbitrary, since F^+ and F^- can always both be multiplied by any constant and still satisfy all the properties that define them. A simple choice is

$$F^-(k) = \frac{1}{F^+(-k)}, \tag{21}$$

which implies that

$$F^-(0) = \sqrt{f_0}. \tag{22}$$

For many purposes the limiting forms of F for large and small k are sufficient. One can write

$$F(k) \approx \sqrt{(k^2 f_\infty^2 + f_0^2)}, \tag{23}$$

where f_∞ and f_0 are chosen to get the two limits right for real k. Then one has

$$F^-(k) \approx \sqrt{(f_0 + ikf_\infty)}, \quad F^+(k) \approx \frac{1}{\sqrt{(f_0 - ikf_\infty)}}. \tag{24}$$

These forms are adequate for studying behavior near the tip, where only large k is important, and interpolate sensibly to large distances, although on the wrong length scale.

4. Comparison of Two Methods for Finding Energy Balance

It has been well established since the 1970s that the energy flux into a plate undergoing fracture depends only upon the *instantaneous* tip velocity, the geometry in which the crack travels, and the loading (Nilsson, 1974; Freund, 1973, 1974; Kostrov, 1974; Kanninen and Popelar, 1985, p. 216; Willis, 1990). This conclusion is surprising in light of earlier speculation (Willis, 1967). In fact the conclusion is false for our geometry. We will show that for problem (A) one is led to a contradiction with elementary principles of elasticity.

Consider problem (A). In this case, the plate stores a potential energy W per length to the right of the crack, which is released as the crack travels. No external forces do work on the plate, so the change in energy of the plate is given exclusively by the flux out the tip, and the total energy of the plate is

$$E = E_0 - Wl(t). \tag{25}$$

It is tempting but incorrect to suppose that the flux through the crack tip is independent of acceleration, so that Eq. (25) applies even if the crack executes an arbitrary motion described by a function $l(t)$. Consider an infinite strip in which a crack moves quasi-statically until it reaches some length l_f. Compare with a second identical trip in which an identical crack begins moving quasi-statically, but then accelerates and reaches length l_f moving at nine-tenths of the Rayleigh wave speed. According to Eq. (25), the two plates have the same energy at the moment the two cracks are of length l_f, although one crack is moving quickly, and the other is not. It is easy to see that the moving crack must have greater energy than the static crack. Solutions of static linear elastic problems can be found as strain configurations which minimize the energy functional (LOVE, 1944, p. 171). Therefore the potential energy of the moving crack must be greater than or equal to the potential energy of the static crack. In addition, the kinetic energy of the moving crack must be positive. Therefore, the moving crack must have greater energy than the static crack. Equation (25) must be false for accelerating cracks.

The local analysis of Freund is not appropriate to our geometry. FREUND (1972b) warns against viewing the independence of stress intensity and acceleration as "a more general result than it really is". His results, as well as those of KOSTROV (1974) and NILSSON (1974) apply to cracks in infinite plates, not to strips. Motion of a crack in a strip involves the reflection of waves from the top and bottom surfaces of the strip back onto the crack tip. In order to deduce an equation of motion for a crack in a strip, it is necessary to turn to other methods for finding energy balance.

5. CALCULATION OF ENERGY, AND DYNAMICAL CONSIDERATIONS

We now carry out an exact computation of the energy of a crack in a strip, using Eq. (2). We first need to find the kinetic energy,

$$K = \rho w v^2 \int_0^1 \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x \sum_\alpha \left(\frac{\partial u_\alpha}{\partial x}\right)^2$$

$$= \rho w v^2 \int_0^1 \mathrm{d}y \int \frac{\mathrm{d}k}{2\pi} \sum_\alpha k^2 u_\alpha(k) u_\alpha(-k). \tag{26}$$

The important point is that one does not need actually to carry out the Wiener–Hopf decomposition in order to perform these integrals. The expressions are rather long, but one can see by inspection of (9) and (11) that every term in them is the product of known functions, two of the coefficients $a_{cl}(-k)$, and $a_{st}(k)$, let us say, and finally the product

$$u_y^0(k)u_y^0(-k) = \frac{\delta F^-(0)}{ikF^-(k)} \frac{\delta F^-(0)}{-ikF^-(-k)} = \frac{\delta^2 f_0}{k^2 F(k)},$$

using (17), (20) and (21). Thus one need not know $F^+(k)$ and $F^-(k)$ at all to compute $K(v)$.

The computation is straightforward from this point. One can do the integrals over y in (26) analytically, but the integral over k must be done numerically. The expressions are of sufficient length that we simply record the answer in the appendix; we have used MAXIMA to do all the algebra. It is necessary to work out separate expressions for six separate regimes: for small, moderate and large k, both for general v, and separately for small v.

With the kinetic energy $K(v)$ known, we now return to the virial theorem, Eq. (2). The surface term in (2) differs for problems (B) and (A), although $K(v)$ is the same in either case. For problem (B) the surface integral is evaluated as follows: if the plate we are working with is truly infinite, the surface term diverges. We will therefore assume that the plate is extremely long but finite, and that the length of plate to the left of the crack tip is $l(t)$, the crack length, where l is extremely large compared to the height of the plate. The only nonzero contributions to the surface integral come from the two crack surfaces; the contributions are equal, and sum to

$$-w\int_{-l(t)}^{0} \mathrm{d}x\, u_y(x,0)\sigma_{yy}(x,0).$$

This may be rewritten as

$$wl(t)\sigma_{\infty}\delta - w\int_{-l(t)}^{0} \mathrm{d}x[u_y(x,0)\sigma_{yy}(x,0)+\delta\sigma_{\infty}]$$

$$= wl(t)\sigma_{\infty}\delta - w\int_{-\infty}^{\infty} \mathrm{d}x[u_y(x,0)\sigma_{yy}(x,0)+\theta(-x)\delta\sigma_{\infty}],$$

dropping terms of order $1/l$. The function $\theta(x)$ is the step function; recall also that $u_y(x,0)$ vanishes for $x > 0$. Converting to the Fourier transform, and using the definition of W, we have

$$-w\lim_{k\to 0}\left[u_y(k,0)(-\sigma_{\infty})+\frac{\delta\sigma_{\infty}}{ik}\right]+Wl(t).$$

One finds that

$$-w\lim_{k\to 0}\left[u_y(k,0)(-\sigma_{\infty})+\frac{\delta\sigma_{\infty}}{ik}\right] = iW\frac{\partial \ln F^{-}(k)}{\partial k}\bigg|_{k=0} \equiv \Sigma(v). \tag{27}$$

Therefore the energy surrounding the crack, Eq. (2), may be written for problem (B) as

$$E = 2K(v)+Wl(t)+\Sigma(v). \tag{28}$$

For problem (A), which is of greater physical interest, we again make the strip finite in length, setting the uncracked region to the right of the crack tip at $t = 0$ to be of length L. Then the surface integrals from the top and bottom of the plate sum to

$$\lim_{k\to 0} w\delta\left[\sigma_{yy}(k,1)+\frac{\sigma_{\infty}}{ik}\right]+w\delta\sigma_{\infty}(L-l(t)) = -\Sigma(v)+w\delta\sigma_{\infty}(L-l(t)). \tag{29}$$

In order to evaluate $\Sigma(v)$, we have decomposed $F(k)$ into $F^{-}(k)/F^{+}(k)$. We have carried out the decomposition numerically in the following way: consider

$$g(l) = \frac{F(bl/\sqrt{(1-l^2)})}{\sqrt{f_\infty^2(b^2l^2/1-l^2)+f_0^2}}.$$

g is a bounded non-oscillatory function on $[-1, 1]$. The constant b should be chosen, for any velocity v, so as to make g as well behaved as possible. A bad choice of b leads all of the structure in g to be compressed either near the origin or near ± 1. We choose b so that the two symmetric minima of g lie at $l = \pm 1/2$. We next approximate g to any desired accuracy as a sum of Chebyshev polynomials, using 61 in practice to achieve one part in 10^6 accuracy. We then find all the complex roots $l_0, \ldots, l_n$ of g. The only numerical difficulty arises in the fact that polishing roots can cause roots spuriously to collapse upon their neighbors. Following advice given by PRESS *et al.* (1988) in Section 9.5, we find that Maehly's procedure cures the problem completely. One should always test that $F^-(0) = \sqrt{f_0}$ to numerical accuracy (KNAUSS, 1967; RICE, 1967). Because $F(k)$ is real and an even function of k, if l_1 is a root, then $-l_1$ is a root as well. We can write g in the form

$$g(l) = g_0 \prod_{i=1}^{m} (l^2 - l_i^2) \Rightarrow F(k) = \sqrt{f_\infty^2 k^2 + f_0^2} \prod_{i=1}^{m} \left[\left(\frac{k^2}{b^2+k^2} - l_i^2 \right) \bigg/ (1 - l_i^2) \right].$$

The Wiener–Hopf decomposition of a function of this form may be carried out by inspection:

$$F^-(k) = \sqrt{if_\infty k + f_0} \prod_{i=1}^{m} \left(\frac{k - b(l_i/\sqrt{(1-l_i^2)})}{k - ib} \right). \tag{30}$$

In order to check the accuracy of (30), and the validity of our approximation scheme, we have verified that, when sufficiently large numbers of Chebyshev polynomials are used (>55, if accuracy of 1 part in 10^6 is desired), the results are independent of the precise number of polynomials. We have also compared with the calculational scheme used by KNAUSS (1966), and found that our procedure agrees with his up to a certain k_u, which increases with the number of roots of $F(k)$ one includes, after which Knauss' calculation of $F^-(k)$ becomes inaccurate (RICE, 1967).

Choosing as a fracture criterion that energy $2\gamma w$ is needed to create crack surface, we have finally that

$$2K(v) - \Sigma(v) + \Sigma(0) + l(2\gamma w - W) = 0, \tag{31}$$

defining the crack to be of length zero when its velocity is zero. This equation constitutes an equation of motion for the crack whenever its acceleration is sufficiently slow that the crack may be regarded as moving between a succession of steady states. For low velocities, the crack accelerates uniformly, since $K(v) \sim \Sigma(v) - \Sigma(0) \sim v^2$ for small v. The maximum possible velocity is the Rayleigh wave speed. Notice also that, if we take the time derivative of (31), we find the energy flux into the tip of the crack to be given by

$$vW - 2\frac{\partial K(v)}{\partial v}\dot{v} + \frac{\partial \Sigma(v)}{\partial v}\dot{v}.$$

We see explicitly how energy flux into the tip depends upon acceleration, when the acceleration is slow enough that there is ample time for sound waves to communicate with the top and bottom of the strip.

When scaled properly, the equation of motion (31) depends only upon the Poisson ratio $\nu = 1 - 2c_t^2/c_l^2$. In order to accomplish the scaling, define a dimensionless energy function

$$e(v/c_t) = \frac{2K(v) - \Sigma(v) + \Sigma(0)}{\delta^2 w \rho c_t^2}.$$

Because our expressions for the energy are so elaborate, we have obtained a simple approximate formula which gives $e(v/c_t)$ to accuracy of five parts in 10^4. For Poisson ratio $\nu = 0.35$, the Rayleigh wave speed is $v_R = 0.9209\ldots$ We find that for this Poisson ratio, and for $v \in [0, 0.915]$,

$$2K(v) - \Sigma(v) + \Sigma(0) \approx \frac{v^2}{[1-(v/v_R)^2]^{0.84}} P[2(v/v_R) - 1].$$

Here

$$P(x) = \sum_{k=1}^{12} c_k T_{k-1}(x) - c_1/2$$

is a sum of Chebyshev polynomials [our conventions are set by PRESS *et al.* (1988, Section 5.7)]; the coefficients c_k are given in Table 1. We have not analyzed the nature of the singularity near v_R in detail; it cannot be fit by any power law, but our approximate expression is adequate in the given interval.

Also defining a dimensionless crack length (recall that we have already set the strip to unit width)

TABLE 1. *Coefficients for a polynomial fit to energy as a function of velocity*

c_1	1.14661
c_2	0.0754189
c_3	0.0346785
c_4	0.0127196
c_5	0.00529015
c_6	0.0019317
c_7	6.42542×10^{-4}
c_8	2.64133×10^{-5}
c_9	-9.43933×10^{-5}
c_{10}	-1.82672×10^{-4}
c_{11}	-1.05454×10^{-4}
c_{12}	-7.59577×10^{-5}

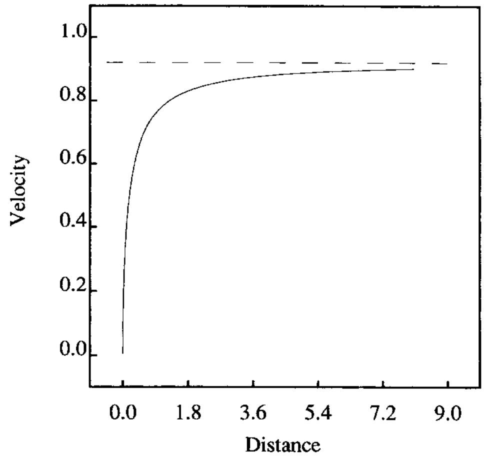

FIG. 2. Crack velocity relative to the transverse wave speed c_t shown as a function of dimensionless distance $\hat{l}$ for Poisson ratio $v = 0.35$. The Rayleigh wave speed is indicated by the dashed line.

$$\hat{l} = l\frac{W-2\gamma w}{\delta^2 w\rho c_t^2} = l\frac{c_l^2}{c_t^2}\left[1-\frac{2\gamma w}{W}\right],$$

and a dimensionless time

$$\hat{t} = tc_t\frac{c_l^2}{c_t^2}\left[1-\frac{2\gamma w}{W}\right],$$

one finds the scaled equation of motion

$$\frac{\mathrm{d}\hat{l}}{\mathrm{d}\hat{t}} = e^{-1}(\hat{l}). \tag{32}$$

The solution of this equation, for $v = 0.35$, is shown in Fig. 2. The adiabatic condition for this solution to be valid is that the characteristic time of acceleration should be much greater than the time required by sound to travel across the width of the strip. Since the strip has width 1, the condition is that

$$\frac{c_l^2}{c_t^2}\left[1-\frac{2\gamma w}{W}\right] \ll 1.$$

We have taken γ to be independent of v for simplicity, but one may easily extend the results to include velocity dependence.

6. DISCUSSION

Numerous problems prevent a comparison of this result with experiment. Most severe is the fact that the stress needed to initiate a crack is always greater than the stress needed for continued propagation. Thus actual cracks do not accelerate from

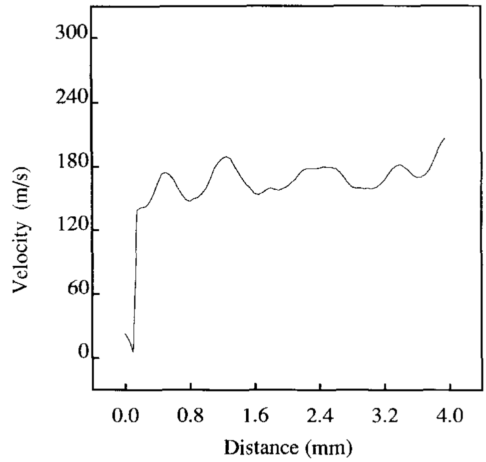

FIG. 3. Experimental measurement of the velocity of a crack, courtesy of J. FINEBERG, S. GROSS, and H. SWINNEY, showing the large jump in velocity at the onset of dynamic fracture in PMMA. The Rayleigh wave speed is around 910 m/s. The oscillations in the crack velocity are all on the scale of experimental error.

zero velocity, but jump nearly instantaneously to a fraction of the limiting velocity; in experiments conducted in our laboratory, the initial velocity jump is on the order of a tenth of the Rayleigh wave speed (Fig. 3). Thus the assumption that the crack accelerate adiabatically is violated from the outset. Additional problems are created by the effects of stress waves which reflect from system boundaries back onto the tip after the crack snaps into motion, and by dynamical instabilities of the tip which continually cause it to deviate from the assumption of adiabatic acceleration. Nonetheless, we believe the formulae we have derived to be valuable because they constitute the most complete and correct account of which we are aware describing how the crack would move in the absence of these instabilities. We are searching for experimental devices by which to diminish the initial velocity jump, while simultaneously extending the theory to account for it. Of course, as is well known (RAVI-CHANDAR and KNAUSS, 1984a–d; KANNINEN and POPELAR, 1985), the current formulae predict that the crack would have the Rayleigh wave speed as a limiting velocity, while real cracks do not approach that speed. This fact was our central motivation for the present work, as we hope to use our calculations as a base from which to study dynamical instabilities.

In comments directed at establishing a simple physical notion of how cracks propagate, ESHELBY (1969) made the remark that "the tip exhibits no inertia". Certainly nothing prevents the crack tip from changing velocity arbitrarily. Yet the notion which has evolved from this remark, that the crack as a whole has no inertia, is incorrect. We have seen that a slowly accelerating crack develops kinetic and potential fields about it which require energy to change. A crack tip has no inertia but steady-state crack motion does. We are unsure how far this conclusion should be taken. It may be that the actual conditions under which cracks propagate are sufficiently far removed from steady-state motion that the assumptions underlying simple dynamical calculations will have to be revised altogether.

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Appendix

In this appendix, we record our expression for the kinetic energy of a crack in a strip, using notion from Section 3

$$K(v) = \int_{-\infty}^{\infty} \mathrm{d}k \frac{1}{2\pi F(k)} \rho v^2 \rho c_l^2$$

$$\times \left[\begin{array}{l} \dfrac{\alpha\beta\begin{pmatrix} (\beta^2-1)(D^2-C^2) \\ +(\alpha^2-1)(B^2-A^2) \end{pmatrix} k - \alpha(\beta^2+1)CD + 4A\alpha\beta C - A(\alpha^2+1)B\beta}{2\alpha\beta k} \\ + e^{\alpha k} + \begin{pmatrix} \dfrac{(B+A)(D-C)e^{-\beta k}}{2k} \\ -\dfrac{(B+A)(D+C)e^{\beta k}}{2k} \end{pmatrix} + \begin{pmatrix} \dfrac{(B-A)(D+C)e^{\beta k}}{2k} \\ -\dfrac{(B-A)(D-C)e^{-\beta k}}{2k} \end{pmatrix} e^{-\alpha k} \\ + \dfrac{(\beta^2+1)(D+C)^2 e^{2\beta k}}{8\beta k} - \dfrac{(\beta^2+1)(D-C)^2 e^{-2\beta k}}{8\beta k} \\ + \dfrac{(\alpha^2+1)(B+A)^2 e^{2\alpha k}}{8\alpha k} - \dfrac{(\alpha^2+1)(B-A)^2 e^{-2\alpha k}}{8\alpha k} \end{array} \right],$$

where

$$A = \frac{(\beta^2+1)(s_\alpha s_\beta - \alpha\beta c_\alpha c_\beta + 2\alpha\beta/(\beta^2+1))}{\alpha(\beta^2-1)(\alpha\beta s_\alpha c_\beta - c_\alpha s_\beta)},$$

$$B = \frac{\beta^2+1}{\alpha(\beta^2-1)},$$

$$C = \frac{2}{\beta^2-1},$$

$$D = \frac{2\alpha\beta s_\alpha s_\beta - 2c_\alpha c_\beta + \beta^2 + 1}{(\beta-1)(\beta+1)(c_\alpha s_\beta - \alpha\beta s_\alpha c_\beta)}$$

and $F(k)$ is given by Eq. (14).