I. Introduction

The interface between nonlinear dynamics and quantum mechanics has become an active area of research in recent years. This emerging field, known as quantum chaos, has focused on the quantum behavior of systems that are chaotic in the classical limit (Haake, 1991; Reichl, 1992). One of the key predictions for Hamiltonian systems with a discrete spectrum is a quantum suppression of chaos. A classical ensemble of particles in a chaotic phase space should execute a random walk, leading to diffusive growth in momentum and position (Tabor, 1989; Lichtenberg and Lieberman, 1991). A system of quantum particles, in contrast, was predicted by Casati et al. (1979) to diffuse in phase space only for a limited time, following the classical prediction. The diffusion is predicted to cease after the "quantum break time" due to quantum interference, and then settle into an exponential distribution. This striking effect, known as dynamical localization, has stimulated a great deal of interest and discussion since it was first predicted. Dynamical localization was predicted to occur in a wide range of systems, and was
also shown by Fishman et al. (1982) to be closely related to Anderson localization, a suppression of electronic conduction in a disordered metal at low temperature (Anderson, 1958; Lee and Ramakrishnan, 1985).

Experimental observation of dynamical localization requires a globally chaotic (classical) phase space, because diffusion can also be restricted by residual stable islands, and by classical boundaries according to the Kolmogorov, Arnol’d, Moser (KAM) theorem (Tabor, 1989; Lichtenberg and Lieberman, 1991). The duration of the experiment must exceed the quantum break time, so that quantum effects can be manifested. Finally, the system must be sufficiently isolated from the environment that quantum interference effects can persist.

Experimental investigations of quantum chaos started with the study of microwave ionization of hydrogen by Bayfield and Koch (1974). Since those first pioneering experiments, atomic physics has become an important experimental and theoretical testing ground for quantum chaos, with the emphasis on strongly driven or strongly perturbed systems such as Rydberg atoms in strong microwave, or magnetic fields (Delande and Buchleitner, 1994). In particular, suppression of ionization in the microwave experiments was attributed to dynamical localization (Galvez et al., 1988; Bayfield et al., 1989; Blümel et al., 1989). The advantage of these systems is that evolution is nonlinear and Hamiltonian. The one-dimensional model is reasonably accurate for the chosen Rydberg states and linear polarization of the microwave field. One complication is the presence of stable structures in phase space. It also has not been possible to measure the time evolution in phase space in order to observe the initial diffusion followed by dynamical localization after the quantum break time. These limitations created strong motivation to find new experimental systems that can be used to investigate dynamical localization as well as other problems in quantum chaos.

This chapter is a review of our work on the motion of atoms in time-dependent potentials. In particular, we study momentum distributions of ultra-cold atoms that are exposed to time-dependent one-dimensional dipole forces. As we will show, the typical potentials are highly nonlinear, so that the classical equations of motion can become chaotic. Because dissipation can be made negligibly small in this system, quantum effects can become important. This work was originally motivated by a theoretical proposal of Graham et al. (1992), and has evolved over the last few years into a series of experiments on dynamical localization and quantum chaos (Collins, 1995).

The organization of this chapter is as follows. In Section II we give a theoretical background on atomic motion in a far-detuned dipole potential, and provide a classical analysis of the potential in terms of nonlinear resonances. In Section III we describe the general experimental approach. In Section IV we discuss the cross-over from classical stability to chaos via the mechanism of resonance overlap, and describe our experimental tests of this phenomena. In Section V we introduce the δ-kicked rotor, a paradigm for classical and quantum chaos, and describe our experimental realization, leading to the observation of the quantum break time,
dynamical localization, and quantum resonances. In Section VI we describe our experiments with a modulated standing wave that also exhibits dynamical localization, illustrating the universal nature of this phenomena. Finally, in Section VII we outline some directions for future work in this emerging field.

II. Two-Level Atoms in a Standing-Wave Potential

Because this work deals with momentum transfer from light to atoms, it is important to review some basic concepts. The relevant unit of momentum is one-photon recoil ($\hbar k_L$). This is the momentum change that an atom experiences when it scatters a single photon, and leads to a velocity change of 3 cm/s for the case of sodium atoms. How does an atom scatter light? The most familiar process is absorption, followed by spontaneous emission. The absorption is from the laser beam, however, the emission is in three dimensions. This process is very important in laser cooling and trapping, but is not desirable for coherent evolution. The probability of spontaneous scattering is proportional to the laser intensity, and inversely proportional to the square of the detuning of the laser from atomic resonance (Cohen-Tannoudji, 1992). This scaling law is valid when the detuning is much larger than the natural linewidth of the atomic transition, and when the intensity is low enough (or the interaction time is sufficiently short). The desired process for atom optics (Adams et al., 1994) is a stimulated scattering, where the atom remains in the ground state, and coherently scatters the photon in the direction of the laser beam. In a single beam (traveling wave) the atom scatters in the forward direction, and there is no net momentum transfer. However, in a standing wave of light created by the superposition of two counterpropagating beams, the atom can also back-scatter. This process leads to a momentum change of two photon recoils. Because the effective dipole potential that the atom experiences only scales inversely with detuning, it is possible to make the probability of spontaneous scattering negligible, while still having a substantial dipole potential.

We begin a more detailed analysis by considering a two-level atom of transition frequency $\omega_o$, interacting with a standing wave of near-resonant light. If the standing wave is composed of two counterpropagating beams, each with field amplitude $E_o$ and wavenumber $k_L = 2\pi/k_L = \omega_o/c$, then the atom is exposed to an electric field of the form $\tilde{E}(x, t) = \frac{\gamma}{\hbar} E_o \cos(k_L x) e^{-i\omega_o t}$ and its Hamiltonian in the rotating-wave approximation is given by

$$H(x, p, t) = \frac{p^2}{2M} + \hbar \omega_o |e\rangle \langle e| + [\mu E_o \cos(k_L x) e^{-i\omega_o t}|e\rangle \langle g| + H.c.]$$

(1)

Here $|g\rangle$ and $|e\rangle$ are the ground and excited internal states of the atom, $x$ and $p$ are its center of mass position and momentum, $M$ is its mass, and $\mu$ is the dipole moment coupling the internal states.
Using standard techniques, we obtain two coupled Schrödinger equations for the ground, $\psi_g(x, t)$, and excited, $\psi_e(x, t)$, state amplitudes

$$i\hbar \frac{\partial \psi_g(x, t)}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi_g(x, t)}{\partial x^2} + \frac{\hbar \Omega}{2} \cos(k_L x)\psi_e(x, t)$$

(2)

$$i\hbar \frac{\partial \psi_e(x, t)}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi_e(x, t)}{\partial x^2} + \frac{\hbar \Omega}{2} \cos(k_L x)\psi_g(x, t) + \hbar \delta_L \psi_g(x, t)$$

(3)

where $\Omega/2 = \mu E_o/\hbar$ is the Rabi frequency of an atom interacting with just one of the light beams. Note that spontaneous emission from the excited state is neglected; this approximation is valid for the case of large detunings $\delta_L = \omega_e - \omega_L$ from the atomic resonance. The large detuning also permits an adiabatic elimination of the excited state amplitude, resulting in a single equation for the ground state amplitude

$$i\hbar \frac{\partial \psi_g}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi_g}{\partial x^2} + \frac{\hbar \Omega^2}{4\delta_L} \cos^2(k_L x)\psi_g$$

(4)

The wavefunction of the now “structureless” atom obeys a Schrödinger equation with a one-dimensional Hamiltonian

$$H(x, p, t) = \frac{p^2}{2M} - V_e \cos 2k_L x$$

(5)

The potential has a period of one-half the optical wavelength and an amplitude $V_e$ that is proportional to the intensity of the standing wave and inversely proportional to its detuning:

$$V_e = \frac{\hbar \Omega^2}{8\delta_L}$$

(6)

$$= \frac{2}{3} \frac{\hbar (\Gamma/2)^2}{\delta_L} \frac{l}{I_{sat}}$$

Here $\Gamma$ is the linewidth of the transition and $\mu$ is its dipole matrix element. $I$ is the intensity of each of the beams comprising the standing wave and $I_{sat} = \pi \hbar \omega_c \Gamma/3 \lambda_L^2$ is the saturation intensity for the transition ($I_{sat} = 6$ mW/cm$^2$ for the case of sodium atoms). Equation (6) was derived for a standing wave composed of two counterpropagating beams of equal intensities. If the two beams are not perfectly matched the potential amplitude is still given by this equation, with $l$ taken as the geometric mean of the two intensities.

The classical analysis of Eq. (5) is the same as for a pendulum or rotor, except that the conjugate variables are position and momentum rather than angle and angular momentum. Such a potential is also known as a nonlinear resonance, and is a fundamental building block of Hamiltonian nonlinear dynamics. It is important
to stress at this point that we are interested in the full nonlinear behavior, and will not limit our analysis to the bottom of the wells, where a harmonic approximation is valid.

A convenient representation of phase space is obtained by evolving the classical equations of motion in time, with some period $T$. This results in a graphical representation known as a Poincaré surface of section, which is shown for the standing wave in Fig. 1. The position coordinate is shown for one period of the standing wave, and the momentum is in units of the two-photon recoil. There is a stable fixed point at the center corresponding to the bottom of the potential well, and an unstable fixed point corresponding to the top. The closed orbits represent oscillatory motion for particles with energy less than the total well depth $2V_0$, whereas the continuing paths describe the unconfined motion of higher-energy
particles. Classically, motion is restricted along the lines shown in Fig. 1 according to the KAM theorem (Tabor, 1989; Lichtenberg and Lieberman, 1991). Note that for large momenta the lines become almost straight, indicating free-particle motion. At smaller momenta, the lines bulge out near the center of the well, corresponding to the particle speeding up as it approaches the bottom of the well.

This chapter emphasizes quantum-classical correspondence, which is especially appropriate for deep wells. It is worth mentioning that for weak wells, this system is naturally described in terms of Bloch bands, a concept that is most familiar from condensed matter physics. Indeed, quantum transport in optical lattices has become an active area of research with recent experiments on Bloch oscillations, Wannier-Stark ladders, and tunneling. Unlike the periodic potentials in condensed matter systems, the atom optics system is effectively free from dissipation mechanisms such as phonon scattering and imperfections in the lattice periodicity. Experimental work on this subject was recently reviewed by Raizen et al. (1997a), and is continuing in our laboratory as well as in other groups. In yet another direction, time-dependent dipole potentials have also found applications in atomic interferometry (Srirangarajan et al., 1996; Cahn et al., 1997), and manipulation of atomic wavepackets is a rapidly growing area.

To address the problem of quantum chaos, we must go beyond the pendulum or stationary standing wave. The connection between atom optics and quantum chaos was first recognized by Graham et al. (1992), who proposed that dynamical localization could be observed in the momentum transfer of ultra-cold atoms in a phase-modulated standing wave of light. More generally, as shown next, quantum chaos can be studied by adding to the one-dimensional Hamiltonian an explicit time dependence. This can be accomplished, for example with a time-dependent amplitude or phase of the standing wave. The electric field of the standing wave then takes the form \( \vec{E}(x, t) = f(E, F_1(t) \cos\{k_F[x - F_2(t)]\} e^{-i\omega_0 t} + c.c. \). The time scales for these controls ranged between \(-25\) ns (the response time of our optical modulators) and milliseconds (the duration of the experiments). The amplitude and phase modulations were therefore slow compared to the parameters \( \omega_0 \) and \( \delta_0 \) relevant to the derivation of Eq. (5), so they change that equation by simply modifying the amplitude and phase of the sinusoidal potential. The generic time-dependent potential is thus

\[
H(x, p, t) = \frac{p^2}{2M} + V_0 F_{\text{amp}}(t) \cos[2k_F x - F_\phi(t)]
\]  

(7)

For simulations and theoretical analyses it is helpful to write Eq. (7) in dimensionless units. We take \( x_\mu = 1/2k_F \) to be the basic unit of distance, so the dimensionless variable \( \phi = x/x_\mu = 2k_Fx \) is a measure of the atom's position along the standing-wave axis. Depending on the time dependence of the interaction, an appropriate time scale \( t_\mu \) is chosen as the unit of time; the variable \( \tau = t/t_\mu \) is then a measure of time in this unit. The atomic momentum is scaled accordingly into
the dimensionless variable $\rho = p_t / M x_u = p 2 k t_u / M$. This transformation preserves the form of Hamilton's equations with a new (dimensionless) Hamiltonian

$$\mathcal{H}(\phi, \rho, \tau) = H(x, p, t) \cdot t_u^2 / M x_u^2 = H \cdot 8 \omega t_u^2 / \hbar.$$  

With this scaling, Eq. (7) can be written in the dimensionless form

$$\mathcal{H}(\phi, \rho, \tau) = \frac{\rho^2}{2} + k f_{\text{amp}}(\tau) \cos[\phi - f_{\text{ph}}(\tau)] \tag{8}$$

The scaled potential amplitude is $k = V_0 \cdot 8 \omega t_u^2 / h$. In these transformed variables, the Schrödinger equation in the position representation becomes

$$i k \frac{\partial}{\partial \tau} \Psi(\phi, \tau) = \left[ -\frac{k^2}{2} \frac{\partial^2}{\partial \phi^2} + k f_{\text{amp}}(\tau) \cos[\phi - f_{\text{ph}}(\tau)] \right] \Psi(\phi, \tau) \tag{9}$$

Here the dimensionless parameter $k$ depends on the temporal scaling used in the transformation

$$k = 8 \omega t_u \tag{10}$$

In the transformation outlined here, the commutation relation between momentum and position becomes $[\phi, \rho] = i k$. Thus $k$ is a measure of the quantum resolution in the transformed phase-space. Another general note on this transformation concerns the measure of the atomic momentum. Because an atom interacts with a near-resonant standing wave, its momentum can be changed by stimulated scattering of photons in the two counterpropagating beams. If a photon is scattered from one of these beams back into the same beam, the result is no net change in the atom's momentum. However, if the atom scatters a photon from one of the beams into the other, the net change in its momentum is two photon recoils. The atom can thus exchange momentum with the standing wave only in units of $2 \hbar k_L$. In the transformed, dimensionless units, this quantity is

$$\frac{\rho}{2 \hbar k_L} = \frac{\rho}{k} \tag{11}$$

For a sample of atoms initially confined to a momentum distribution narrower than one recoil, the discreteness of the momentum transfer would result in a ladder of equally spaced momentum states. In our experiments the initial momentum distributions were significantly wider than two recoils, so the observed final momenta had smooth distributions rather than discrete structures.

## III. Experimental Method

The experimental study of momentum transfer in time-dependent interactions consists of three main components: initial conditions, interaction potential, and
measurement of atomic momentum. The initial distribution should ideally be narrow in position and momentum, and should be sufficiently dilute so that atom–atom interactions can be neglected. The time-dependent potential should be one-dimensional (for simplicity), with full control over the amplitude and phase. In addition, noise and coupling to the environment must be minimized to enable the study of quantum effects. Finally, the measurement of final momenta after the interaction should have high sensitivity and accuracy. Using techniques of laser cooling and trapping it is possible to realize all these conditions.

A schematic of the experimental setup is shown in Fig. 2. Our initial conditions are a sample of ultra-cold sodium atoms, which are trapped and laser-cooled in a magneto-optic trap (MOT) (Chu, 1991; Cohen-Tannoudji, 1992). The atoms are contained in an ultra-high vacuum glass cell at room temperature. The cell is attached to a larger stainless steel chamber, which includes a 20 l/s ion pump. The source of atoms is a small sodium ampoule contained in a copper tube that is attached to the chamber. The ampoule was crushed to expose the sodium to the rest...
of the chamber. Although the partial pressure of sodium at room temperature is below \(10^{-10}\) torr, there are enough atoms in the low-velocity tail of the velocity distribution that can be trapped. The trap is formed using three pairs of counter-propagating, circularly polarized laser beams (2.0 cm beam diameter), which intersect in the middle of the glass cell, together with a magnetic field gradient that is provided by current-carrying wires arranged in an anti-Helmholz configuration. This configuration is now fairly standard and is used in many laboratories. These beams originate from a dye laser that is locked 20 MHz to the low frequency (red) side of the \((3S_{1/2}, F = 2) \rightarrow (3P_{1/2}, F = 3)\) sodium transition at 589 nm.

Approximately \(10^5\) atoms are trapped in a cloud that has an rms size of 0.15 mm, with an rms momentum spread of \(4.6 \hbar \ell\). This distribution would be represented in the Poincaré surface of section of Fig. 1 as a band that is narrow in momentum, but uniform in position on the scale of a standing wave. The potential is provided by a second dye laser that is tuned typically 5 GHz from resonance (both red and blue detunings were used with no difference in the experimental results). The output of this laser is aligned through a fast acousto-optic modulator (25 ns rise time), which is driven by a pulse generator. This device controls the laser intensity in time. The beam is then spatially filtered to ensure a Gaussian intensity profile, and is centered on the atoms, with a \(1/e^2\) field waist of \(w_0 \sim 1.9\) mm. For the single-pulse and kicked rotor experiments (Sections IV and V, respectively) the beam was retro-reflected from a mirror outside the vacuum chamber to create a standing wave, as shown in Fig. 3(a). For the modulated standing wave experiments (Section VI) a more complicated setup was used as shown in Fig. 3(b).

To what extent is Eq. (7) a good representation of a sodium atom exposed to an optical standing wave in the laboratory? The two-level atom and rotating-wave approximations are well justified for this optical-frequency transition. The adiabatic elimination of the excited-state amplitude is appropriate for the values of detuning and intensity that were used in the experiments. The detuning was also large compared to the linewidth \(\Gamma\) and to the recoil shift frequency \(\omega_r\). For the sodium \(D_2\) transition, the values for these quantities are

\[
\Gamma \over 2\pi = 10 \text{ MHz}
\]

and

\[
\omega_r \over 2\pi = \frac{1}{2\pi} \frac{\hbar k_i^2}{2M} = 25 \text{ kHz}
\]

The atoms were prepared in a particular hyperfine ground state (in some experiments they were prepared in the \(F = 2\) state, whereas in others they were optically
pumped to the $F = 1$ state); however, they were not optically pumped into a particular Zeeman sublevel. This was not a problem because with linearly polarized light and the large detuning, all the $m_F$ sublevels experienced the same potential.
The one-dimensional nature of Eq. (7) comes from the assumption that the laser beams have spatially uniform transverse profiles. In these experiments the width of the atomic cloud during the illumination by the standing wave was small compared to the width of the laser profile.

The detection of momentum is accomplished by allowing the atoms to drift in the dark for a controlled duration after the interaction with the standing wave. Their motion is frozen by turning on the optical trapping beams in zero magnetic field to form “optical molasses” (Chu, 1991; Cohen-Tannoudji, 1992). The motion of the atoms is overdamped, and for short times (tens of ms) their motion is negligible. The position of the atoms is then recorded via their fluorescence signal on a charged coupled device (CCD) and the time of flight is used to convert position into momentum. The entire sequence of the experiment is computer controlled.

In Fig. 4, typical two-dimensional images of atomic fluorescence are shown. In Fig. 4(a) the initial MOT was released, and the motion was frozen after a 2-ms free-drift time. This enables a measurement of the initial momentum distribution. The distribution of momentum in Fig. 4(a) is Gaussian in both the horizontal and vertical directions. The vertical direction is integrated to give a one-dimensional distribution as shown in Fig. 5(a). In Fig. 4(b), the atoms were exposed to a particular time-dependent potential. The vertical distribution remains Gaussian, but the horizontal distribution becomes exponentially localized due to the interaction potential, as shown in Fig. 5(b). The significance of the lineshape and other characteristics are analyzed next.
Fig. 5. One-dimensional atomic momentum distributions. They were obtained by integrating along the vertical axes of the two-dimensional distributions in the previous figure. The horizontal axes are in units of two recoils, and the vertical axes show fluorescence intensity on a logarithmic scale. (a) Initial thermal distribution with no interaction. (b) Localized distribution after interaction with the potential. The characteristic exponential lineshape is discussed in the text.

IV. Single Pulse Interaction

The simplest time-dependent potential that we can impose is the turning on and off of the standing-wave intensity. In the context of atom optics, this type of time-dependent interaction occurs, for example, whenever an atomic beam passes
through a standing wave of light. Diffraction from a standing wave was first studied by Martin et al. (1987) where the emphasis was on the two regimes of Raman-Nath and Bragg scattering. The time dependence in those cases was determined by the atoms traversing the Gaussian profile of the standing wave. Initial theoretical models assumed a sudden turn on/off of the standing wave, and it was believed that the details of the temporal profile merely led to an overall correction term.

We now reexamine this simple process from the standpoint of classical nonlinear dynamics and find a very different answer. As a first approach to this problem, one expects that for slow turn on/off the evolution is adiabatic. The conditions for adiabaticity are very clear for linear potentials such as the harmonic oscillator. The difficulty with nonlinear potentials is that there are many time scales, so the conditions for adiabaticity must be examined much more carefully. We show that in this case the temporal profile can have important dynamical consequences and find that the intermediate regime between the sudden and adiabatic can lead to mixed phase space and chaos.

To analyze this problem in more detail, we assume a generic time dependent potential

$$V(x, t) = V_c F(t) \cos 2k_p x$$

(14)

For the case of atomic beam diffraction (Martin et al., 1987), \( F(t) = \exp(-t^2/\tau^2) \). We consider here the case \( F(t) = \sin^2 \pi t/T_s \), which is turned on for a single period \( T_s \).

This Hamiltonian can be expanded as

$$H = p^2/2M - V_c \sin^2 \pi t/T_s \cos 2k_p x$$

$$= p^2/2M - (V_c/2) \{ \cos 2k_p x$$

$$- (\cos 2k_p (x - v_m t) + \cos 2k_p (x + v_m t))/2 \}$$

(15)

where \( v_m = \lambda_k/2T_s \). The effective interaction is that of a stationary wave with two counterpropagating waves moving at \( \pm v_m \). Classically, there are now three resonance zones each of width proportional to \( \sqrt{V_c} \), and separation in momentum proportional to \( T_s^{-1} \). The Poincaré surface of section for this Hamiltonian is shown in Fig. 6. Keeping \( V_c \) constant and increasing \( T_s \) leads to the overlap of these isolated resonances and a subsequent destruction of the KAM surfaces. This mechanism for cross-over from stability to chaos was formulated by Walker and Ford (1969) and by Chirikov (1979). In this case particle motion is no longer restricted to move along the lines of each isolated resonance. The resulting phase space is generally mixed, with islands of stability surrounded by regions of chaos. This leads to diffusion in certain regions of phase space, and confinement in others. An example of a surface of section in that case is shown in Fig. 7 for parameters that are accessible experimentally. Relative to the atomic diffraction experiments of
Fig. 6. Poincaré surface of section for the \( \sin^2 \) potential. In this case there are three isolated resonances.

Martin et al. (1987), this regime requires a combination of deep wells with significant atomic motion (on the scale of the standing-wave period), and is clearly outside the limiting cases of Raman-Nath or Bragg.

To experimentally determine the threshold \( \tau_{cr} \) for overlap, we must distinguish the momentum growth associated with spreading within the primary resonance from diffusion that can occur after resonance overlap. This is accomplished by contrasting the momentum transfer from the potential due to a standing wave of fixed amplitude

\[
V'(x) = (V/2) \cos(2kLx)
\]  

(16)
for duration $T_s$ with

$$V(x, t) = (V_s/2) \left[ \cos 2k_L(x - (x - v_m t)) + \cos 2k_L(x + v_m t) \right]$$

resulting from the $\sin^2$ amplitude modulated standing wave. The experimental setup is shown in Fig. 3(a). The key to the interpretation of the experimental results is the realization that prior to resonance overlap $V'(x)$ and $V(x, t)$ should give the same result. After overlap of the resonances, $V(x, t)$ will result in significantly larger momentum transfer than $V'(x)$. The experimental results in Fig. 8(b) show the rms momentum for both cases as a function of pulse duration (rise and fall times of 25 ns are included in the square pulse duration). These agree well

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**Fig. 7.** Poincaré surface of section for the $\sin^2$ potential after resonance overlap has occurred. There is a bounded region of global chaos.
Fig. 8. (a) RMS momentum computed from a classical simulation for \(\sin^2\) (solid line) and square (dashed line) pulses. (b) RMS momentum from experimentally measured distributions for \(\sin^2\) (solid) and square (open) pulses for the same conditions as (a). (c) RMS momentum computed from a quantum simulation for \(\sin^2\) (solid line) and square (dashed line) pulses for the same conditions as (a). The threshold estimated from resonance overlap is indicated by the arrow. A clear deviation occurs at a pulse duration close to the predicted value (Robinson et al., 1996).

with classical numerical simulations shown in Fig. 8(a) as well as the estimated resonance overlap threshold (Robinson et al., 1996).

The predicted quantum behavior is shown in Fig. 8(c). For the case of \(V'(x)\), we find close agreement with the classical simulations and with the experiment. This is an interesting result in its own right, because the coherent oscillations that occur for short times are seen in the experiment with a large ensemble of indepen-
dent atoms and in the quantum simulation, which uses a single wavepacket approach. For the case of $V(x, t)$ there is also good agreement between the three cases over the entire range of pulse times; however, the quantum widths are slightly lower than the corresponding classical values near the large peak in the rms width. Although this difference is too small to be of quantitative significance, it is nevertheless the precursor for differences in quantum and classical behavior that can occur when the classical dynamics are globally chaotic. These differences, which form the basis for the study of quantum chaos, are the focus of the next experiments we discuss.

V. Kicked Rotor

A. Introduction

The classical $\delta$-kicked rotor, or the equivalent standard mapping, is a textbook paradigm for Hamiltonian chaos (Lichtenberg and Lieberman, 1991). A mechanical realization would be an arm rotating about a pivot point. The rotation is free, except for sudden impulses that are applied periodically. The Hamiltonian for the problem is given by

$$\mathcal{H} = \frac{p^2}{2} + K \cos \phi \sum_{n=-\infty}^{\infty} \delta(\tau - n) \quad (18)$$

The evolution consists of resonant-kicks that are equally spaced in time, with free motion in between. The quantity $K$ is called the stochasticity parameter, and is the standard control parameter for this system. As $K$ is increased, the size of each resonant-kick grows. Beyond a threshold value of $K \approx 4$ it has been shown that phase space is globally chaotic (Reichl, 1992). The chaos is due to the fact that the magnitude of each kick depends on the angle of the rotor at that moment and the nonlinearity of the potential. Note that, in contrast, a kicked harmonic oscillator cannot be chaotic because it is a linear system. It is intuitively clear that for a given kick strength, motion can become chaotic if the duration between kicks becomes long enough. This is because after one kick the particle has time to evolve to a completely different point in phase space before the next kick occurs.

The quantum version of this problem has played an equally important role for the field of quantum chaos since the pioneering work of Casati et al. (1979) and Chirikov (1979). In particular, dynamical localization was predicted to occur for the kicked rotor and detailed scaling laws were derived. Although this model may seem unique, many physical systems can be mapped locally onto the kicked rotor, so that it is actually a universal paradigm system.

To observe dynamical localization in an experimental realization of the kicked rotor, we turned the standing wave on and off in a series of $N$ short pulses with period $T$. This system differs from the ideal kicked rotor in two ways. The first
difference is that the conjugate variables here are position and momentum instead of angle and angular momentum, so that strictly speaking our system consists of kicked particles. The second is that the pulses have finite duration instead of being δ-kicks. The effect of finite pulse duration was also considered by Blümel et al. (1986) in the context of molecular rotation excitation. The first distinction might seem problematic, because there is a natural quantization of angular momentum, in contrast to a continuum of momentum states for a free particle. In our system, however, the quantization of momentum is imposed by the periodicity of the wells, so that the momentum kicks must occur in units of two recoils. The initial distribution, on the other hand, can be continuously distributed over different momentum states, providing averaging of diffusion and localization. The effects of finite pulse duration are analyzed next, but we note here that if the atoms do not move significantly compared to the spatial period during a pulse, this system is an excellent approximation of the δ-kicked rotor.

Atomic motion in this case can be described by the Hamiltonian of Eq. (7) with $F_{\text{opp}} = \sum_{n=1}^{N} F(t - nT)$ and $F_{\text{pp}} = 0$,

$$H = \frac{p^2}{2M} + V_0 \cos(2k_f x) \sum_{n=1}^{N} F(t - nT)$$

(19)

Here the function $F(t)$ is a narrow pulse in time centered at $t = 0$ that modulates the intensity of the standing wave. The sum in this equation represents the periodic pulsing of the standing-wave amplitude by multiplying $V_0$ with a value in the range $0 \leq F(t) \leq 1$.

The optical arrangement for the experiment was described in Section III and illustrated in Fig. 3(a). The fast acousto-optic modulator (AOM) provided the amplitude modulation of the standing wave to form the pulse train $\sum F(t)$. This modulator had a 10–90% rise and fall time of 25 ns. The number of pulses and pulse period were computer controlled with a arbitrary waveform generator. A sample trace of the pulse profiles recorded on photodiode PD1 is shown in Fig. 9.

With the scaling introduced in Section II and the unit of time taken to be $T$, the period of the pulse train, the Hamiltonian for this system becomes

$$\mathcal{H} = \frac{p^2}{2} + K \cos \phi \sum_{n=1}^{N} f(\tau - n)$$

(20)

The train of δ-functions in Eq. (18) has been replaced here by a series of normalized pulses $f(\tau) = F(\tau T) / \int_{-\infty}^{\infty} F(\tau T) d\tau$. Note that the scaled variable $\tau = t/T$ measures time in units of the pulse period. As described earlier, $\phi = 2k_f x$ is a measure of an atom’s displacement along the standing wave axis and $p$ is its momentum in units of $2\hbar k_f / k$.

Aside from the temporal profile of the pulses, all the experimental parameters that determine the classical evolution of this system are combined into one quantity, the stochasticity parameter $K$. As we will see, the quantum evolution depends
Fig. 9. Digitized temporal profile of the pulse train measured on a fast photo-diode. The vertical axis represents the total power in both beams of the standing wave. \( f(t) \) and \( \Omega_{q} \) are derived from this scan (Moore et al., 1995).

Additionally on the parameter \( k \). These two dimensionless quantities thus characterize the dynamics of Eq. (20). In terms of the physical parameters of Eq. (19), they are

\[
K = 8V_{o}\alpha T_{p}\omega_{p}/\hbar
\]

\[
k = 8\omega_{p}T
\]

Here \( t_{p} \) is the FWHM duration of each pulse, and \( \alpha = \int_{-\infty}^{\infty} F(t) dt/t_{p} \) is a shape factor that characterizes the integrated power for a particular pulse profile: it is the ratio of the energy in a single pulse to the energy of a square pulse with the same amplitude and duration. For a train of square pulses, \( \alpha = 1 \); for Gaussian pulses, \( \alpha = (\pi/4 \ln 2)^{1/2} \approx 1.06 \). For the roughly square pulses used in our experiments, \( \alpha \) was within a few percent of unity.

B. Classical Analysis

Atoms with low velocities do not move significantly during the pulse, so their classical motion can be described by a map. By integrating Hamilton’s equations of motion over one period, we obtain the change in an atom’s displacement and momentum:

\[
\Delta \phi = \int_{\Phi_{n-1/2}}^{\Phi_{n+1/2}} dt \rho = \rho
\]

\[
\Delta \rho = \int_{\rho_{n-1/2}}^{\rho_{n+1/2}} dt K \sin \phi \sum f(\tau - n) = K \sin \phi
\]
The discretization of these relations is the classical map,

\[ \phi_{n+1} = \phi_n + \rho_{n+1} \]

\[ \rho_{n+1} = \rho_n + K \sin \phi_n \]

that is known as the “standard” map (Reichl, 1992). For small values of \( K \), the phase space of this system shows bounded motion with regions of local chaos. Global stochasticity occurs for values of \( K \) greater than \( \sim 1 \), and widespread chaos appears at \( K > 4 \), leading to unbounded motion in phase space. Correlations between kicks in the spatial variable \( \phi \) can be ignored for large values of \( K \), so this map can be iterated to estimate the diffusion constant. After \( N \) kicks, the expected growth in the square of the momentum is

\[ \langle (\rho_N - \rho_0)^2 \rangle = K^2 \sum_{n=0}^{N-1} \left\langle \sin^2 \phi_n \right\rangle + K^2 \sum_{n \neq n'} \left\langle \sin \phi_n \sin \phi_{n'} \right\rangle \]

\[ = \frac{K^2}{2} N \]

The diffusion in momentum is thus

\[ \langle \rho^2 \rangle = DN, \quad \text{with} \quad D \equiv \frac{K^2}{2} \]  

Note that this description, which follows from the discretization into the standard map, requires the duration of the pulses to be short. To understand the effects of a finite pulse-width, consider the case where the pulse profile \( f(\tau) \) is Gaussian with an rms width \( \tau_0 \). In the limit of a large number of kicks \( N \), the potential in Eq. (20) can be expanded into a Fourier series:

\[ \mathcal{V} = \rho^2 + K \cos \phi \sum_{m=-\infty}^{\infty} e^{im2\pi \tau} e^{-(m2\pi \tau_0)^2/2} \]

\[ = \rho^2 + \sum_{m=-\infty}^{\infty} K_m \cos(\phi - m2\pi \tau) \]

with

\[ K_m = K \exp[-(m2\pi \tau_0)^2/2] \]

The nonlinear resonances are located (according to the stationary phase condition) at \( \rho = d\phi/d\tau = m2\pi \). This expansion is similar to the resonance structure of the \( \delta \)-kicked rotor, in which the \( K_m \) are constant for all values of \( m \). In Eq. (28), however, the widths of successive resonances fall off because of the exponential term in the effective stochasticity parameter \( K_m \). This fall-off is governed by the pulse profile; the result of Eq. (28) was derived for the case of a Gaussian pulse shape, but in general \( K_m \) is given by the Fourier coefficients of the periodic pulse train.

The nonzero pulse widths thus lead to a finite number of significant resonances in the classical dynamics, which in turn limits the diffusion that results from over-
lapping resonances to a band in momentum. The width of this band can be made arbitrarily large by decreasing the pulse duration and increasing the well depth, thereby approaching the $\delta$-function pulse result. This can be seen in the result just derived. In the limiting case of $\tau_0 \rightarrow 0$ with $K$ fixed (infinitesimal pulse width and very large well depth), we recover the resonance structure expected for the $\delta$-function limit in Eq. (18): $K_m = K$. In the experiment, the pulse width only needs to be small enough that the band of diffusion is significantly wider than the range of final momenta and that the effective diffusion constant $K_m$ is approximately uniform over this range.

An example of the bounded region of chaos that arises from the finite pulse duration is illustrated by the classical phase portrait shown in Fig. 10, for typical

![Poincaré surface of section for the pulsed system using a train of Gaussians to represent the experimental sequence. The integrated area under a single pulse is taken to be the same as in the experiment. The standing wave has a spatial rms value of $\Omega_{\text{rms}}/2\pi = 75.6$ MHz, $T = 1.58$ $\mu$s, and $a = 0.027$, leading to $K = 11.6$. Note that a small intensity variation due to spatial overlap of atoms and laser profile results in a somewhat smaller $K$ than that at peak field (Moore et al., 1995).]
experimental parameters. The central region of momentum in this phase portrait is in very close correspondence with the δ-kicked rotor model with \( K = 11.6 \). This stochasticity parameter is well beyond the threshold for global chaos.

The boundary in momentum can also be understood using the concept of an impulse. If the atomic motion is negligible while the pulse is on, the momentum transfer occurs as an impulse, changing the momentum of the atom without significantly affecting its position. Atoms with a sufficiently large velocity, however, can move over several periods of the potential while the pulse is on. The impulse for these fast atoms is thus averaged to zero, and acceleration to larger velocities is inhibited. The result is a momentum boundary that can be pushed out by making each pulse shorter.

Classically, then, the atoms are expected to diffuse in momentum until they reach the momentum boundary that results from the finite pulse width. Equation (27) indicates that the energy of the system \( \frac{1}{2}(\frac{p^2}{2 \hbar k_f})^2 \) thus grows linearly in time. In terms of the number of pulses \( N \), this energy is

\[
\frac{p^2}{2 \hbar k_f} = \frac{1}{2} \left( \frac{p}{k} \right)^2 = \frac{1}{2k^2} \frac{K^2}{2} N
\]

C. QUANTUM ANALYSIS

This system can be expected to exhibit quantum behaviors that are very different from those predicted classically. A qualitative atom-optics picture of the kicked rotor is that of an atom passing through a series of \( N \) diffraction gratings and then forming an interference pattern. The entire device can be seen as a multistage atomic interferometer, and is an extension of the three grating interferometer proposed by Chebotayev et al. (1985). Each diffraction grating represents a kick, followed by free evolution between the gratings. From this picture it is clear that this is a manifestly quantum system and the final pattern is determined by complicated interference of amplitudes. From that standpoint, it is perhaps surprising that for a small number of gratings before the “break time,” the resulting interference pattern appears “classical.” We now discuss two phenomena that are predicted to occur in the kicked rotor, namely dynamical localization and quantum resonances.

Dynamical localization is the quantum suppression of chaotic diffusion, which is thought to occur in many physical systems but is most cleanly studied here. Quantum resonances are a quantum feature particular to the δ-kicked rotor.

A quantum analysis of this system starts with the Schrödinger equation, Eq. (9). For the pulsed modulation of Eq. (20), this becomes

\[
i k \frac{\partial}{\partial \tau} \Psi(\phi, \tau) = \left[ -\frac{k^2}{2} \frac{\partial^2}{\partial \phi^2} + K \cos \phi \sum_n f(\tau - n) \right] \Psi(\phi, \tau)
\]

The periodic time dependence of the potential implies that the orthogonal solutions to this equation are time-dependent Floquet states. This system has been
studied extensively in the ideal case of \( f(\tau) = \delta(\tau) \) with an infinite train of kicks \( (n = 0, \pm 1, \pm 2, \ldots) \) (Casati et al., 1979). An analysis of this system by Chirikov et al. (1981) shows that this system diffuses classically only for short times during which the discrete nature of the Floquet states is not resolved. As shown by Fishman et al. (1982), Eq. (31) can be transformed into the form of a tight-binding model of condensed-matter physics. An analysis of that system indicates that the Floquet states of Eq. (31) are discrete and exponentially localized in momentum. Because these states form a complete basis for the system, the initial condition of an atom in the experiments can be expanded in a basis of Floquet states. Subsequent diffusion is limited to values of momentum covered by those states that overlap with the initial conditions of the experiment. If the initial conditions are significantly narrow in momentum, the energy of the system should grow linearly with the number of kicks \( N \), in agreement with the classical prediction in Eq. (30), until a "quantum break time" \( N^* \). After this time, the momentum distribution approaches that of the Floquet states that constituted the initial conditions, and the linear growth of energy is curtailed. This phenomenon is known as dynamical localization.

The Floquet states are characterized by a "localization length" \( \xi \) with \( |\Psi(\rho/k)|^2 \sim \exp(-|\rho/k|/\xi) \). The momentum distribution then has a 1/e half-width given by \( p^*/2\hbar k = p^*/k = \xi \), where \( \xi \) is the average localization length of the Floquet states (Reichl, 1992). The number of Floquet states that overlap the initial condition (and therefore the number of Floquet states in the final state) is roughly \( \xi \), so the average energy spacing between states is \( \Delta \omega \sim 1/\xi \). The quantum break time is the point after which the evolution reflects the discreteness of the energy spectrum, hence \( N^* \Delta \omega \sim 1 \), or \( N^* = \xi \). By combining these estimates with Eq. (30), we see that \( \xi \) is proportional to \( K^2/2k^2 \). The constant of proportionality has been determined numerically to be \( \frac{1}{2} \) (Shepelyansky, 1986), and the localization length is thus

\[
\frac{p^*}{2\hbar k} = \xi = \frac{K^2}{4k^2}
\]  

(32)

In our experiments we derive the rms momentum from the measured lineshapes, because its definition applies as well to the prelocalized Gaussian distributions as to the exponentially localized ones. For an exponential distribution, this quantity is larger than the localization length by a factor of \( \sqrt{2} \):

\[
\frac{p^*_{\text{RMS}}}{2\hbar k} = \sqrt{2} \cdot \frac{p^*}{2\hbar k} = \frac{K^2}{2\sqrt{2}k^2}
\]  

(33)

Because \( \xi \) is also a measure of the number of kicks before diffusion is limited by dynamical localization, we have for the quantum break time
\[ N^* = \bar{\xi} = \frac{K^2}{4k^2} \tag{34} \]

An inherent assumption in the derivation of Eqs. (32–33) is the lack of structure in the phase space of the system. Small residual islands of stability, however, do persist even for values of \( K > 4 \). This structure introduces the dynamics dependence on the location of the initial conditions in phase space. Nonetheless, this analysis provides a useful estimate of the localization length and the quantum break time.

D. EXPERIMENTAL PARAMETERS

It is important to consider these last two relations in choosing experimental parameters. In order for a localized distribution to be observable, \( p^* \) must be significantly smaller than the region enclosed by the classical boundary. Thus there is a constraint between the duration of the kicks (parameterized by its FWHM value \( t_p \)) and the localization length. As previously described, the simplest estimate for this condition requires that the distance traveled by a particle during a pulse be much less than a period of the standing wave: \( p_{\text{RMS}}^* t_p / M \ll \lambda_L / 2 \).

A better estimate comes from Eq. (29), which indicates the effective stochasticity parameter for a particle with momentum \( p/2\hbar k_L = m2\pi / k \) in a train of Gaussian pulses with FWHM duration \( t_p = \sqrt{8 \ln 2} \tau_{\text{RMS}} T \). From this equation we see that the effective stochasticity parameter drops below 4, resulting in islands of stability for atoms with momenta greater than \( p_{\text{max}} / 2\hbar k_L = (\ln 2 \ln \xi)^{1/2} / 2\omega_p t_p \). The resonance overlap criterion can provide a more accurate expression, but this estimate is sufficient for determining the range of operating parameters.

The experimental conditions should be chosen so that the localized momentum width \( p_{\text{RMS}}^* \) is much smaller than this limit. On the other hand, the localized momentum distribution needs to be several times wider than the initial distribution so that it can be distinguished from the initial conditions. The atoms in our MOT started with an RMS momentum of \( \sigma_{p0} = -4.6 \hbar k_L \), imposing a lower limit on the localization length of \( p_{\text{RMS}}^* / 2\hbar k_L > \sigma_{p0} / 2\hbar k_L = 2.3 \). Combining these two bounds gives

\[ \frac{\sigma_{p0}}{2\hbar k_L} < \frac{p_{\text{RMS}}^*}{2\hbar k_L} < \sqrt{\ln 2 \ln \frac{K}{4}} / 2\omega_p t_p \tag{35} \]

Another constraint on the localization length comes from its relation to \( N^* \), the number of kicks required for the localization to manifest. This time must be short enough to be observable in the experiment. Indeed, the experiment should continue for a time significantly greater than \( N^* \) so that it is clear that the early period of diffusive growth has ended. An upper limit on the duration of the experiment,
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and therefore on the localization length, comes from the increased probability of spontaneous emission events with longer exposures to the standing wave. Spontaneous emission can randomize the phase of an atomic wavefunction, thereby destroying the coherence necessary for the quantum phenomena under observation. The probability of a spontaneous event during \( N^* \) kicks of duration \( t_p \) is \( 1 - e^{-\gamma_{spont} N^* t_p} \). To preserve the coherent evolution of the atomic sample, we require this probability to be small:

\[
\gamma_{spont} N^* t_p \ll 1
\]  

(36)

Here \( \gamma_{spont} = (V_0, \delta E/\hbar) (\Gamma/2)[(\delta E^2 + (\Gamma/2)^2)^{-1} \) is the probability per unit time for an atom to undergo a spontaneous event, and \( \Gamma/2\pi (\approx 10 \text{ MHz}) \) is the linewidth of the sodium \( D_2 \) transition.

In addition to these constraints relating to the localization length, there are several other restrictions on the experimental parameters. To ensure that the atoms are all subject to the same well depth, the light field cannot vary greatly over the sample of atoms, and thus a lower limit to the beam waist is given by the spatial width of the atomic sample. In our experiments the interaction times were short enough and the initial temperatures cold enough that the sample of atoms did not spread significantly from its initial MOT width of \( \sigma_{w_0} \approx 0.15 \text{ nm (rms)} \), so it was sufficient for the beam waists to be large in comparison to this initial value,

\[
w_0 \gg \sigma_{w_0}
\]  

(37)

In order to observe dynamical localization, the classical phase space must be characterized by extended regions of chaos evident in the classical phase portraits for \( K > 4 \). This requirement set a constraint on the well depth \( V_0 \), the pulse period \( T \) and the pulse duration \( t_p \).

The most important constraint is the maximum power available in the beams that make up the standing wave. Large laser powers help satisfy Eq. (37), because the beams can then be made wide while maintaining the desired intensity at the center of the beam profile. In practice, however, the laser power in each beam \( (P) \) is of course limited and the other experimental control parameters of beam waist \( (w_0) \), detuning \( (\delta E) \), pulse period \( (T) \), and pulse duration \( (t_p) \) must all be chosen to satisfy the criteria enumerated here.

The fact that Eq. (36) can be satisfied is an especially valuable aspect of this experiment. Spontaneous emission is the only significant avenue of energy dissipation from the dilute sample of atoms. By making this dissipation negligible, our system is effectively Hamiltonian. It is interesting to note the features of the system that make this possible. To keep the probability of spontaneous emission small, we take advantage of the different dependencies of the well depth \( V_0 \) and the spontaneous emission rate \( \gamma_{spont} \) on the detuning. The well depth is proportional to the intensity of the standing wave and inversely proportional to the detuning,
\[ V_0 = \frac{2}{3} \frac{h(\Gamma/2)^2}{I} \frac{I}{I_{\text{sat}}} \]

\[ \propto \frac{P}{w_0^2 \delta_\ell} \]  

Although the spontaneous emission rate is also proportional to the intensity, it varies as the inverse square of the detuning.

\[ \gamma_{\text{spont}} = \frac{V_0 \delta_\ell}{\hbar} \frac{\Gamma/2}{\delta_\ell^2 + (\Gamma/2)^2} \]

\[ \propto \frac{P}{w_0^2 \delta_\ell^2} \]  

Within the limits of available laser power, a large detuning can therefore provide negligible spontaneous emission during the experiment without too much loss in the well depth.

In our experiments, each counterpropagating beam typically had a power of \( P = 0.2-0.4 \) W; the waists were in the range of \( w_0 = 1.2-2.2 \) mm, and the detunings from resonance \( \delta_\ell/2\pi \) were between 5 and 10 GHz. These operating conditions led to well depths in the range of \( V_\ell/\hbar = 5-15 \) MHz, and to spontaneous emission probabilities of about 1% per kick. The pulse periods and durations were in the ranges 1-5 \( \mu \)s and 0.05-0.15 \( \mu \)s, respectively.

E. EXPERIMENTAL RESULTS

We subjected the cooled and trapped atoms to a periodically pulsed standing wave and recorded the resulting momentum distributions as described in Section III. To study the temporal evolution of the atomic sample under the influence of the periodic kicks, these experiments were repeated with increasing numbers of kicks \((N)\) with the well depth, pulse period, and pulse duration fixed. These successive measurements provided the momentum distributions at different times in the atomic sample’s evolution. Such a series of measurements is shown in Fig. 11. Here the pulse had a period of \( T = 1.58 \mu s \), and a FWHM duration of \( t_p = 100 \) ns. For these conditions, \( k \) has a value of 2.0. The largest uncertainty in the experimental conditions is in the well depth, \( V_\ell \), which depends on the measurement of the absolute power of the laser beams that make up the standing wave and their spatial profile over the sample of atoms. To within 10\%, the well depth for these data had spatial RMS value of \( V_\ell/\hbar = 9.45 \) MHz. The pulse profile was nearly square, leading to a stochasticity parameter of \( K = 11.6 \), the same value as for the phase portrait in Fig. 10.

The distributions clearly evolve from an initial Gaussian at \( N = 0 \) to an expo-
Fig. 11. Experimental time evolution of the lineshape from the initial Gaussian until the exponentially localized lineshape. The quantum break time is approximately 8 kicks. Fringes in the freezing molasses lead to small asymmetries in some of the measured momentum lineshapes as seen here and in the inset of Fig. 12. The vertical scale is measured in arbitrary units and is linear (Moore et al., 1995).

mentially localized distribution after approximately $N = 8$ kicks. We have measured distributions out until $N = 50$ and find no further significant change. The small peak on the right side of this graph is due to nonuniformities in the detection efficiency. As discussed in Section III, the relative numbers of atoms with different momenta is measured by their fluorescence intensity on a CCD camera. Spatial variations in the MOT beams were due to interference fringes from the chamber windows. This was a minor limitation on the resolution of the momentum measurements, and will be corrected in the future with antireflection coatings on all windows.

The growth of the mean kinetic energy of the atoms as a function of the number of kicks was calculated from the data and is displayed in Fig. 12. It shows an initial diffusive growth until the quantum break time $N^* = 8.4$ kicks, after which dynamical localization is observed (Moore et al., 1995). The solid line in this figure represents the classical diffusion predicted in Eq. (30). The data follow this prediction until the break time. The dashed line in the same figure is the prediction for the energy of the localized distribution from Eq. (33). Though not shown here, classical and quantum calculations both agree with the data over the diffusive regime. After the quantum break time, the classical growth slows slightly due to the fall-off in $K$ predicted by Eq. (29) for nonstationary atoms. The observed distribution would lead to a reduction of only 15% in the stochasticity parameter. Thus
the classically predicted energy would continue to increase diffusively. The measured distributions, however, stop growing as predicted by the quantum analysis.

F. QUANTUM RESONANCES

Between kicks, the atoms undergo free evolution for a fixed duration. The quantum phase accumulated during the free evolution is $e^{-i\Omega^2T/2M\hbar}$. An initial plane wave at $p = 0$ couples to a ladder of states separated by $2\hbar k_f$. For particular pulse periods, the quantum phase for each state in the ladder is a multiple of $2\pi$. A con-
dation known as a “quantum resonance” (Reichl, 1992). More generally, a quantum resonance is predicted when the accumulated phase between kicks is a rational multiple of 2π. We have scanned T from 3.3 μs to 50 μs and find quantum resonances when the quantum phase is an integer multiple of π. For even multiples, the free evolution factor between kicks is unity; for odd multiples, there is a flipping of sign between each kick. Quantum resonances have been studied theoretically, and it was shown that instead of localization, one expects the energy to grow quadratically with time (Casati et al., 1979; Izrailev and Shepelyansky, 1979). This picture, however, is only true for an initial plane wave. We have done a general analysis of the quantum resonances (to be published) and show that for an initial Gaussian wavepacket, or for narrow distributions not centered at p = 0, the final momentum distribution is actually smaller than the exponentially localized one, and settles in after a few kicks.

Our experimental results are shown in Figs. 13 and 14. Ten quantum resonances are found for T ranging between 5 μs (corresponding to a phase shift of π) and 50 μs (10π) in steps of 5 μs. The saturated momentum distribution as a function of T is shown in Fig. 13. The narrower, nonexponential profiles are the resonances between which the exponentially localized profiles are recovered. The time evolution of the distribution at a particular resonance is shown in Fig. 14, from which it is clear that the distribution saturates after very few kicks.

Fig. 13. Experimental observation of quantum resonances as a function of the period of the pulses. The surface plot is constructed from 150 momentum distributions measured, for each T, after 25 kicks. This value of N ensures that the momentum distributions are saturated for the entire range of T shown. On resonance, the profiles are nonexponential and narrower than the localized distributions that appear off-resonance. Note that the vertical scale is linear (Moore et al., 1995).
VI. The Modulated Standing Wave

A. INTRODUCTION

The last experiment described in this chapter was actually the first to be performed in our laboratory and was originally motivated by a proposal of Graham et al. (1992). It is interesting to note that the same interaction Hamiltonian was derived and analyzed in an earlier paper by Graham et al. (1991) for driven Josephson junctions; however, that proposal has not yet been realized experimentally. In retrospect, the modulated system is more subtle than the kicked rotor or the single pulse. In our experiment, atoms are subjected to a standing wave of near-resonant light, where the displacement of the standing wave nodes is modulated at a frequency \( \omega_n \) and with an amplitude \( \Delta L \). Once again, the excited state amplitude is adiabatically eliminated. With this form of the modulation the effective Hamiltonian given in Eq. (7) becomes

\[
H = \frac{p^2}{2M} + V_o \cos[2k_L(x - \Delta L \sin \omega_n t)]
\]  

(40)

Although this Hamiltonian may look somewhat different than the \( \delta \)-kicked rotor, it also displays the phenomenon of dynamical localization, as discussed next.
B. CLASSICAL ANALYSIS

The Hamiltonian of Eq. (40) can be expanded as a sum of nonlinear resonances using a Fourier expansion. By expanding the temporal dependence of the potential, we obtain the resonance structure of the system,

\[
H = \frac{p^2}{2M} + V_c \left[ J_0(\lambda) \cos 2k_L x + J_1(\lambda) \cos 2k_L (x - v_m t) + J_{-1}(\lambda) \cos 2k_L (x + v_m t) + J_2(\lambda) \cos 2k_L (x - 2v_m t) + J_{-2}(\lambda) \cos 2k_L (x + 2v_m t) + \cdots \right] \\
= V_c \sum_{n=-\infty}^{\infty} J_n(\lambda) \cos 2k_L (x - nv_m t)
\]

where \( J_n \) are ordinary Bessel functions, \( v_m = \omega_m / 2k_L \) is the velocity difference between neighboring resonances, and \( \lambda = 2k_L \Delta L \) is the modulation index.

As in the case of the \( \delta \)-kicked rotor, the resonances are located at regular intervals in momentum. The amplitudes of these resonances, however, depend on the modulation index \( \lambda \). The dependence on \( \lambda \) allows this system to be tuned between regimes where the classical dynamics are integrable (for example, \( \lambda = 0 \)) to those in which they are chaotic.

The classical resonances are evenly separated in momentum with central values of

\[
p_n = nMV_m
\]

and widths of

\[
\Delta p_n = 4\sqrt{MV_c |J_n(\lambda)|}
\]

There are substantial resonances only for \( n \leq \lambda \), so for momenta greater than \( M\lambda v_m \) the phase space is characterized by essentially free evolution. These regions of free evolution confine the motion of atoms with small initial momentum to the portion of phase space spanned by the resonances. For certain ranges of \( \lambda \), these resonances overlap, leading to a band of chaos with boundaries in momentum that are proportional to \( \lambda \). A sample of atoms starting with initial conditions within this band will remain within it, confined to momenta in the range \( \pm M\lambda v_m \). A simple estimate of the atomic momentum after a long time is a uniform distribution within these bounds (Graham et al., 1992); such a distribution would have an RMS momentum of

\[
\frac{p_{\text{RMS}}}{2\hbar k_L} = \frac{M\lambda v_m}{\sqrt{3}} \frac{\lambda}{2\hbar k_L} = \frac{\lambda}{\sqrt{3}} \frac{\omega_m}{8\omega_r}
\]
The classical dynamics can also be understood in terms of resonant kicks that occur twice during each modulation period. Consider an atom subjected to the modulated standing wave of Eq. (40). When the standing wave is moving with respect to the atom, the time-averaged force is zero, because the sign of the force changes as the atom goes over “hill and dale” of the periodic potential. Momentum is transferred to the atom primarily when the standing wave is stationary in the rest frame of the atom. These resonant kicks occur twice in each modulation period, but they are not equally spaced in time. The magnitude and direction of the resonant kick depends on where the atom is located within the standing wave at that time.

The calculated variation of the rms momentum width as a function of $\lambda$ is shown in Fig. 15 for $\omega_m/2\pi = 1.3$ MHz and $V/\hbar = 3.1$ MHz. The estimate of Eq. (44) is shown by the straight solid line. For values of $\lambda < 3$, this estimate agrees roughly with an integration of the classical Hamilton's equations by Robinson et al. (1995), (shown in the figure) calculated for an interaction time of 20 $\mu$s. For larger values of $\lambda$, the simulation is lower than the estimate, because in only 20 $\mu$s the initial distribution (with $p_{RMS}/2\hbar_k \approx 2.3$) does not have time to diffuse up to the limit represented by the solid line. The longer-duration classical simulation presented in the figure agrees with the estimate over the entire range of $\lambda$ shown, except for values of $\lambda$ close to 7.0 (explained next). The 20- $\mu$s classical simulation also shows oscillations in the diffusion rate as a function of $\lambda$: peaks in the rms momentum correspond to values of $\lambda$ leading to large diffusion rates, whereas dips indicate slow diffusion.

To understand this variation in diffusion rates, we examine the resonances in Eq. (41). The dependence of the diffusion rate on $\lambda$ is due to oscillations in $J_\ell(\lambda)$, the amplitudes of the resonances. The various resonances grow and shrink as the modulation index $\lambda$ is increased. For certain values of $\lambda$, a resonance can be significantly diminished, or even removed in the case where $\lambda$ is a zero of one of the Bessel functions. As shown in the phase portraits of Fig. 16 (top panel), this variation in the amplitudes of the resonances strongly influences the dynamics of the system. In general, the phase spaces are mixed, with islands of stability surrounded by regions of chaos. Atoms from the initial distribution that are contained within an island remain trapped, whereas those in the chaotic domain can diffuse out to the boundaries. In the case of a diminished resonance, the islands of stability from neighboring resonances might not be destroyed by resonance overlap. This is the case with $\lambda = 3.8$, for which $J_1(\lambda)$ has its first zero. The final momentum spread in this case is governed largely by the surviving island due to the resonance at $p_0 = 0$, and the system is nearly integrable. The stability of this system causes the reduced diffusion shown by the dip in the classical simulation of Fig. 15 at $\lambda = 3.8$. Indeed, all of the dips in this simulation occur at values of $\lambda$ that are near zeros of Bessel functions; the dynamics of the corresponding systems are stabilized by the diminished resonances. This stabilization even affects diffusion in the long-time
Fig. 15. rms momentum width as a function of the modulation amplitude $\lambda$, for $\omega_o/2\pi = 1.3$ MHz and $V_o/\hbar = 3.1$ MHz. Experimental data are denoted by diamonds and have a 10% uncertainty associated with them. The empty diamonds are for an interaction time of 10 $\mu$s and the solid diamonds are for 20 $\mu$s. The straight line denotes the resonant-kick boundary, and the curved line is the prediction of Graham et al. (1992). The three curves indicate numerical simulations. A classical simulation is shown, one for an interaction time of 20 $\mu$s (dot–dash line). The observed data lie well below these curves for some values of $\lambda$. A 20 $\mu$s integration of the Schrödinger equation is also presented for comparison with the corresponding experimental data (heavy dashed line). The heavy solid line is a Floquet-state calculation, and represents the long-time quantum prediction (Robinson et al., 1995).

Classical simulation: for values of $\lambda$ close to 7.0 (the second zero of $J_1(\lambda)$), the initial conditions are trapped in a large island of stability at $p = 0$. For these values of $\lambda$, the diffusion is limited by the width of the island to a value much smaller than that given by the resonant-kick boundary.

Note that the oscillations of the Bessel functions are reflected in the exchange of the location of hyperbolic and elliptic fixed points. At $\lambda = 0$, there is only one resonance in the expansion of Eq. (41) centered at $p_0 = 0$ with an amplitude
\( V_\nu J_\nu(0) = + V_\nu \). The potential minima for this resonance are located in space at even multiples of \( \pi/2k_L \), so the island of stability is centered at \( x = 0 \) in the phase portrait. The phase portraits for \( \lambda = 3 \) and \( \lambda = 3.8 \) also have islands of stability centered in momentum at \( p_0 = 0 \), but the amplitudes for these resonances are negative: \( V_\nu J_\nu(3) = -0.40V_\nu \) and \( V_\nu J_\nu(3.8) = -0.26V_\nu \). The reversal of sign exchanges the location of the potential minima and maxima, so the islands in these portraits are centered in position at \( x = \pi/2k_L \).

Notice also that the overall amplitude of the oscillations decreases as \( \lambda \) is increased due to the reduction in the size of each resonant-kick. This effect can be understood from the impulse approximation, because the maximum classical force is fixed but the time that the standing-wave potential is stationary in the rest frame of the atom is inversely proportional to \( \lambda \). The classical diffusion rate is therefore reduced by increasing \( \lambda \), although the classical saturation value of \( p_{RMS} \) increases with \( \lambda \).

**C. EXPERIMENT**

The experimental realization of the Hamiltonian of Eq. (40) required a somewhat more complicated optical setup for the interaction potential that is illustrated in Fig. 3(b). To modulate the phase of the potential, we vary the phase of one of the two laser beams that make the standing wave. The electro-optic modulator EOM2 in Fig. 3(b) provided this control. For a phase shift of \( \pi \) at 589 nm, this modulator required an applied voltage of \( V_\pi = 271 \) V. By applying an oscillating drive
we modulated the phase of the beam with an amplitude \( \pi V_{EO}/V_r \) and gave the phase of the standing wave a time dependence \( \lambda \sin \omega_m t \), with \( \lambda = 2k L \Delta L = \frac{1}{2} \pi V_{EO}/V_r \). To provide the high voltage required for the phase shifts in this experiment, the signal was stepped-up in a helical resonator (\( V_{pp} = 2V_{EO} = 2400 \text{ V} \), corresponds to \( \lambda = 7 \)). This resonator was designed so that when connected to the capacitive EOM it formed a tuned circuit that had an input impedance of 50 \( \Omega \) at a resonant frequency of 1.3 MHz. The circuit had a Q of 108 and the output voltage across the EOM was stepped-up by a factor of 77. The modulation index was calibrated by measuring the FM sidebands in optical heterodyne, and identifying the appropriate zeros of the Bessel functions.

The main control parameter in the experimental realization was the modulation index, \( \lambda \). The momentum distribution was measured for a range of \( \lambda \) for fixed values of the intensity and detuning. The temporal evolution was not mapped out systematically in these experiments, but the duration was chosen to be long enough to saturate the growth of momenta. The experimental data are shown in Fig. 15 for interaction times of 10 and 20 \( \mu s \). The 20 \( \mu s \) data match the classical simulations well for small values of \( \lambda \) and for values of \( \lambda \) that are close to zeros of Bessel functions. For other values of \( \lambda \), however, the experimentally measured distributions are much narrower than those predicted classically. This reduction is a manifestation of dynamical localization in this system.

The momentum distributions after 20 \( \mu s \) are exponential, as in our kicked rotor experiment. To observe this effect we must ensure that the location of the resonant-kick boundary is much further than the localization length. As this boundary scales linearly with \( \lambda \), we expect to see the appearance of dynamical localization only beyond some value of \( \lambda \). This experimental requirement is similar to the considerations of the classical boundary in the kicked rotor experiments. There, however, the boundary was due to an effective reduction in \( K \) by the motion of an atom over several wells during a single pulse. Here the classical boundary arises from the maximum velocity that can be imparted to an atom by resonant kicks.

Note that for small values of \( \lambda \) the experiment is good agreement with the classical prediction. At \( \lambda = 0 \) the system is integrable and momentum is trivially localized. As \( \lambda \) is increased the phase space becomes chaotic, but growth is limited by the resonant-kick boundary. Our measured momentum distributions (in Fig. 16, bottom panel) are characteristically “boxlike” in this regime (0 \( \leq \lambda \leq 2 \)). This observation is consistent with the picture of a uniform diffusion limited by the boundaries in momentum.

As \( \lambda \) is increased beyond a critical value, there are oscillations in the observed rms momentum. For certain ranges of the modulation index \( \lambda \), the observed values deviate substantially from the classical prediction. These ranges correspond to conditions of large diffusion rates—the peaks in the classical prediction. For these values of \( \lambda \), the classical phase space is predominately chaotic. An example
of the resulting dynamics is shown in Fig. 16 for \( \lambda = 3.0 \). The classically predicted distribution (middle panel) is roughly uniform, but the experimentally observed distribution is exponentially localized (Moore et al., 1994; Robinson et al., 1995); hence the rms value is reduced.

As \( \lambda \) is increased further, the oscillations in the resonance amplitudes lead to phase portraits with large islands of stability, as in the case \( \lambda = 3.8 \). For these values of \( \lambda \), the classical phase space becomes nearly integrable and the measured momentum is close to the classical prediction.

Quantum analyses under the conditions of the experiment as well as an asymptotic (long-time limit) Floquet analysis are shown along with the classical simulations and experimental data in Fig. 16. The predicted distributions from the Floquet analysis are displayed along with the experimentally observed ones in the lower panel of Fig. 16. It is clear that there is good quantitative agreement between experiment and the effective single-particle analysis (Robinson et al., 1995; Bardroff et al., 1995).

Graham et al. (1992) showed that the modulated system can be approximated by the \( \delta \)-kicked rotor. Although this connection is valid in certain parameter regimes, it is important to stress that dynamical localization is not restricted to that model system, and can occur in any chaotic phase space. Even in the \( \delta \)-kicked rotor, which is the paradigm system, the simple scaling laws that relate diffusion rate with localization length are valid only in the limit of asymptotically large stochasticity parameter. For smaller values of \( K \), the residual structures in phase space can modify local behavior. The same is true for the modulated system and is probably a feature of any experimentally accessible system. Our experimental initial conditions average over a band in phase space, yielding average values for diffusion and localization length. A more complete discussion of this point was covered in a recent series exchange of letters (Latka and West, 1995; Raizen et al., 1997a,b; Menenghini et al., 1997; Latka and West, 1997).

VII. Conclusion and Future Directions

In this chapter we reviewed our experiments on dynamical localization with ultracold sodium atoms. There are many interesting questions that can now be addressed experimentally with this system. One direction is to study how dynamical localization may be destroyed by noise or dissipation. This problem has been the topic of a great deal of theoretical work (see for example Ott et al., 1984; Dittrich and Graham, 1987; Fishman and Shepelyansky, 1991). Experiments on microwave ionization of Rydberg atoms studied the effects of amplitude noise, and an increased ionization probability as a function of noise amplitude was observed (Blümel et al., 1989). In our present system of the \( \delta \)-kicked rotor, noise and dissipation could be introduced as amplitude or phase noise. We can also induce spontaneous scattering by illuminating the atoms with a weak resonant beam dur-
ing the coherent evolution. We should then be able to follow the growth of momentum as a function of time for different types and levels of noise, and hopefully gain a better understanding of decoherence in this system. One of the limitations of the current sodium experiment is the boundary in phase space. This becomes especially problematic for studies of delocalization, where momentum should grow substantially beyond the localization length. To overcome this problem, we are building a new experiment based on cesium atoms. The boundary (measured in recoil units) should be pushed out by more than an order of magnitude relative to the sodium case. This should enable a detailed study of the effects of noise and dissipation.

The role of dimensionality on dynamical localization has been studied theoretically in detail. A transition to power-law localization in two dimensions and delocalization in three dimensions was predicted. This could be studied experimentally by introducing several spatial or temporal periodicities in the potential (Casati et al., 1989). The spatial periodicity of the standing wave, for example, can be increased by making the angle between the two beams less than 180°. Incommensurate spatial periods can be superimposed with several far-detuned standing waves at different angles. The standing waves must also be detuned from each other so that cross-interference terms move at a high velocity and are averaged out.

The focus of this work has so far been on cases where the classical phase space is globally chaotic. The more generic situation in nature is a mixed phase space, consisting of islands of stability surrounded by regions of chaos. To study this regime, better initial conditions are needed. We have developed a new method that should enable the preparation of a minimum-uncertainty “box” in phase space, and plan to implement this technique in our cesium experiment. This would enable a detailed study of quantum transport in mixed phase space. Some interesting topics to study would be tunneling from islands of stability, chaos assisted tunneling, and quantum scars (Heller and Tomsovic, 1993).

VIII. Acknowledgments

I would like to thank Fred Moore, John Robinson, Cyrus Bharucha, Kirk Madison, and Steven Wilkinson for their important contributions to these experiments. I would also like to thank Bala Sundaram and Qian Niu for excellent theoretical support. This work was supported by the U.S. Office of Naval Research, the Robert A. Welch Foundation, and the U.S. National Science Foundation.

IX. References

QUANTUM CHAOS WITH COLD ATOMS