See
Predictability and Chaos
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Discovery 13, 27-31 (1993)
Predictability and Chaos

Some phenomena are inherently unpredictable even though chance plays no role and the mathematical rules governing the phenomena are known

The daily newspaper contains a prediction based on the laws of physics: the times of the sunrise and sunset. The same section of the newspaper contains another prediction based on known laws of physics: tomorrow's weather. Why is one prediction extremely accurate while the other is often in error?

We can know the physical laws and still have unpredictable behavior. This is the paradox of chaos. One would think that we simply need to determine the present state of a system and then use the known laws to predict future behavior. This deterministic view of nature, accepted for centuries, was nicely expressed by Pierre Simon de Laplace in 1814:

An intellect which at any given moment knew all the forces that animate Nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit its data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain, and the future just like the past would be present before its eyes.

Erratic phenomena are familiar to all of us—the trajectory of a balloon when it is released, cardiac arrhythmia, the incidence of disease, the weather—but such erratic behavior has generally been thought to arise from fluctuations in the environment. Now we realize that erratic behavior can also arise in the absence of environmental fluctuations.

Chaos: The Exponential Growth of Uncertainties. Laplace's view that the behavior of all entities can be predicted at any time in the future implicitly assumes that uncertainties will grow only slowly in time. If the uncertainty grows exponentially fast, however, after a short time it will be so large that nothing can be said about the subsequent state of the system. This is chaos.

The possibility of the exponential growth of uncertainties in systems was recognized by the French mathematician Henri Poincaré, who said in 1913 "...it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible..." Until recently Poincaré's warning about inherently unpredictable behavior in deterministic systems was regarded by mathematicians as a curiosity and was essentially unknown to laboratory scientists.

Unpredictable behavior is not new in the laboratory, but it is usually attributed to poor laboratory technique or to difficulties in establishing reproducible initial conditions. Now the availability of computers makes it possible to test laboratory data and mathematical models to distinguish between noisy external influences and unpredictable behavior arising from exponential growth of uncertainties.

Chaos and Weather Forecasting. The first concrete example of a system exhibiting the exponential growth of uncertainties was discovered in the early 1960s at the Massachusetts Institute of Technology by Edward Lorenz, a meteorologist interested in weather forecasting. He realized that the available computers were far too small to solve the equations for the atmosphere, so he examined an extremely simple model that served as a caricature of the atmosphere. For some values of the parameters in the model, the computer solutions of his model fluid relaxed to a steady state. For other values of the parameters, however, the behavior never settled down; it just varied...
irregularly as long as it was followed on the computer. One day Lorenz accidentally examined computer solutions for two almost identical initial states, and to his surprise, he found that the difference between the two initial states increased exponentially fast.

Lorenz quickly realized the incredible implications of his discovery. The prevailing view at the time was that as computers become larger and larger and larger, weather forecasts will become better and better and better. Weather prediction is an extremely difficult problem involving rain, clouds, snow, the topography of the earth, the interaction of the oceans and atmosphere, and many other factors. Nonetheless, the equations describing this complex system are basically known. Thus, given a sufficiently large computer, it should be possible to predict the weather far into the future. But Lorenz realized that this was wrong.

Lorenz reasoned as follows. One can imagine making, at an instant of time, perfect measurements, free of any uncertainty, of temperature, velocity, pressure, etc., at every point in the atmosphere and oceans. Then, given these conditions and a sufficiently large computer, the weather could be predicted arbitrarily far into the future. But perfect measurements are impossible. Lorenz recognized that no matter how small the uncertainty in the initial conditions and how large the computer, it still would not be possible to predict the weather accurately far into the future. The fundamental limit is about two weeks. Current forecasts are accurate for at most about six days.

**PeriodDoublingRoute to Chaos.**

The next major development in the theory of chaos was made in the late 1970s by a physicist at Los Alamos, Mitchell Feigenbaum, who investigated the simplest possible abstract nonlinear model that yields time-dependent behavior. He found that for some range of values of the model control parameter, his system oscillated with a period one (for example, one year). As the parameter was increased, a critical value was reached at which the period of oscillation doubled—the behavior repeated with a period of two years. With further increase of the parameter, the period doubled again, to four years. And on and on, more and more doublings, but the amount that the externally controlled parameter had to be changed between successive doublings decreased very rapidly, and quickly the limit of this sequence was reached—a state with an infinite period (two multiplied times itself an infinite number of times). Beyond that point the solutions of the model exhibited the defining property of chaos: two states, initially almost identical, diverged apart exponentially fast. Chaotic solutions never repeat, no matter how long they are observed. The solutions just vary erratically in time, as if there was no underlying rule (hence the term chaos). Nevertheless, the behavior is precisely determined by the mathematical model.

The period doubling route to chaotic behavior was interesting in itself, but what was much more remarkable was that several other model systems were examined and also found to give a period doubling sequence leading to chaos. Not only that, the approach to chaos was described by the same mathematical relations! Feigenbaum’s amazing discovery was submitted for publication in physics journals but was rejected because it was “not physics,” just a computer exercise. Now we realize that period doubling and chaos often occur in real systems.

**Chaos in the Laboratory.** Lorenz and Feigenbaum studied extremely simple abstract models. In 1975 Jerry Gollub of Haverford College and I began a search for chaos in a laboratory physical system, flow between concentric cylinders with the inner cylinder rotating while the outer cylinder remains at rest (Figure 1). This problem was suggested by Newton in his *Principia* (1687) but was not studied in laboratory experiments until nearly two centuries later. In the past decade the problem has been the subject of a thousand research papers, yet many flow phenomena exhibited by this system remain poorly understood.

At small rotation rates of the cylinder every particle in the flow rotates in a circle that is concentric with the axis of the cylinders. When the rotation rate is increased, a critical value is reached at which the flow spontaneously self-organizes into donut-shaped rolls that encircle the inner cylinder and are stacked along the cylinder axis, as shown in Figure 1a. The flow has structure that is periodic in the axial direction, but the fluid velocity measured at any point in the flow is independent of time. At a larger, well-defined rotation rate of the cylinder, the rolls develop waves, as Figure 1b illustrates. The velocity at any point in the flow oscillates periodically in time. At a yet larger rotation rate a second wave appears on the donuts. This flow, like that at lower rotation rates, is accurately predicted by Newton’s laws, expressed in the form applicable to continuous fluids rather than to individual particles. By predictable we mean that if the velocity is measured at several points in the flow, the future velocity can be predicted with high accuracy. With further increase in cylinder rotation rate, there is a transition to a flow that varies erratically in time, as Figure 1c shows. In the 1980s mathematical tools became available to measure the rate of

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growth of uncertainties, and we showed that this erratic flow is chaotic—uncertainties in the velocity grow exponentially fast.

**Period Doubling and Universality.**

The first observation in the laboratory of Feigenbaum’s period doubling sequence leading to chaos was made in the late 1970s by a Parisian physicist, Albert Libchaber, who examined convection in a fluid contained between parallel plates heated from below. Soon thereafter we observed this type of behavior in experiments on a chemical reaction, as will now be described. Reacting chemicals are fed continuously into a tank, and the chemicals produced by the reaction, along with unreacted chemicals, are removed at the same rate as the feed. For small flow rate, the concentrations of the chemicals do not vary in time, but beyond a critical value of the flow rate, the chemical concentrations spontaneously begin to oscillate [Figure 2a]. The oscillation period is about ninety seconds. At a larger well-defined flow rate, the chemical concentrations still oscillate but the maximum concentration is slightly different for every other oscillation [Figure 2b]. The time required for the concentration to return to the original maximum value is now 180 seconds—the period has doubled.
With further increase in the feed rate, the period doubles again (Figure 2c). Now four oscillations are required before the behavior repeats. And at a yet higher, well-defined feed rate the period doubles again! With further increases in feed rate, the period doubles again and again. The end of the infinite sequence of doublings of the period marks the onset of chaos. The concentrations still oscillate, but in an apparently erratic manner, never repeating (Figure 2d).

Period doubling sequences have been found in a wide variety of systems, including lasers, sound waves, and vibrating machinery. What is the relation between Feigenbaum's simple mathematical models and these complex physical systems? It is not yet known why systems even as different as a beating heart and a dripping faucet exhibit markedly similar dynamics—period doubling and chaos. This amazing universality in the dynamics of systems that are governed by quite different physical laws excites scientists, engineers, and mathematicians and is a major reason for the lure of chaos.

**Strange Attractors.** Graphs of the chemical concentrations as a function of time succinctly describe periodic behavior (Figure 2), but it is hard to interpret a graph of a chaotically varying quantity that never repeats the same shape. A better way to describe the behavior of chaotic systems is in terms of the motion of a single point in an appropriate abstract space. This point draws a portrait of the dynamics. This is the ultimate abstraction: any system, whether as simple as a pendulum or as complex as the entire universe, can in principle be represented at an instant of time by a single point in an abstract space. This point moves as the state of the system changes (for example, as the concentrations in the chemical reaction vary). If the behavior is periodic, the point in this abstract space must return to its original position in a time equal to the period.

Thus the portrait for the periodic state in Figure 2a is a closed loop, as shown in Figure 3a. The portraits for the period 2 and 4 states are, respectively, loops with two and four turns. The power of these abstract portraits is illustrated by the one in Figure 3b for the chaotic reaction. The term **strange attractor** has been coined to describe the portraits for chaotic systems. The strange attractor in Figure 3b was discovered in our laboratory by J.C. Roux, a visitor from Bordeaux, France. The behavior of a chaotic system cannot be predicted far into the future, yet the strange attractor clearly has a beautiful underlying structure. **There is order in chaos.** This structure indicates that the system is deterministic: the behavior is governed by rules involving no element of chance.

The degree of unpredictability can be determined from an analysis of the strange attractor. Suppose that two initial states of the reacting chemical system are nearly identical; they would be represented by two nearby points in Figure 3b. The rate of loss of predictability is given by the rate at which these two points separate. Nearby points in Figure 3b separate exponentially fast, their separation almost doubling every two complete trips around the attractor. No matter how close the two points were initially, that is, no matter how nearly identical the two systems were, after only a few trips around the attractor the pair of points become widely separated—in other words, the systems will have become quite different in behavior.

**Zebra Stripes and Leopard Spots.** Chaos research for the past decade has concerned mainly the problem of time dependence: how does a system evolve in time? Even in the flow of fluid between concentric cylinders, research has concerned mainly different kinds of time dependence (periodic, multiply periodic, chaotic). Now scientists are turning to the more difficult
problem of pattern formation: how does a system evolve in space as well as time?

Our laboratory is investigating the formation of patterns in chemical systems that are not stirred. Stirring, as in the experiments in Figures 2 and 3, eliminates any variation of the concentrations in space, but in the absence of stirring, concentrations can vary in space. The possibility of the spontaneous formation of chemical spatial patterns was first proposed by the brilliant British mathematician Alan Turing, who is best known for his universal computer, the “Turing machine.” Turing’s 1952 paper “The chemical basis for morphogenesis” suggested that spatial patterns in chemical systems could explain animal coat patterns such as stripes on a zebra or spots on a leopard.

Turing’s prediction has served as the basis for extensive work in theoretical biology on patterns in plants and animals, but his hypothesis remained untested until recent experiments in Bordeaux and by the UT Austin group yielded the first laboratory Turing patterns. Striped and hexagonal Turing patterns obtained in experiments by Qi Ouyang in our laboratory are illustrated in Figures 4a and 4b. Although the experiments provide clear evidence of the Turing instability, Turing’s conjecture of a relationship between chemical and biological patterns remains to be demonstrated.

Turing predicted the transition from a uniform state [no pattern] to a steady chemical pattern. Beyond the transition to steady patterns such as those shown in Figures 4a and 4b, we have found a transition to unsteady patterns that are chaotic in both space and time (Figure 4c is an example). Chaotic patterns are well-known in fluid dynamics but are new in chemistry.

The Future. Naturalists have long focused on the balance of nature or the rhythms of nature, not sporadic events such as epidemics or famines. Economists have emphasized the balance between supply and demand, not the Great Depression or Black Monday. Now we realize that erratic behavior—chaos—can occur even in well-controlled systems with no external fluctuating influences. No matter how long the behavior of a system is observed, and no matter how accurate the measurements of its initial state, it may still be impossible to predict the long-term future behavior. The behavior for many situations is accurately predictable, as in the case of the sunrise and sunset, and uncertainties in the specification of earth’s motion grow slowly, not exponentially fast. But for many other cases, such as the weather, uncertainties grow exponentially fast, thus making long-term prediction impossible.

The development of this subject from the fringes of science to the mainstream was the subject of a 1987 best seller, Chaos, Making a New Science by James Gleick. While the persuasiveness of chaos is now widely accepted, it is far from being understood. Even if we know the equations and the initial state of a system, we still cannot even say (except for a few special cases) whether the behavior will be predictable or chaotic unless we examine the problem on a computer or in a laboratory. Nor is the mathematical universality of chaos understood. Why is chaos in systems as different as a flowing fluid, a vibrating machine, and an electrical circuit described by the same mathematical relations? Thus many challenges remain for future research.