

# Quantum Chaos of Bogoliubov Waves for a Bose-Einstein Condensate in Stadium Billiards

Chuanwei Zhang,<sup>1,2</sup> Jie Liu,<sup>3,1</sup> Mark G. Raizen,<sup>1,2</sup> and Qian Niu<sup>1</sup>

<sup>1</sup>*Department of Physics, The University of Texas, Austin, Texas 78712-1081, USA*

<sup>2</sup>*Center for Nonlinear Dynamics, The University of Texas, Austin, Texas 78712-1081, USA*

<sup>3</sup>*Institute of Applied Physics and Computational Mathematics, P.O. Box 100088, Beijing, Peoples Republic of China*

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We investigate the possibility of quantum (or wave) chaos for the Bogoliubov excitations of a Bose-Einstein condensate in billiards. Because of the mean field interaction in the condensate, the Bogoliubov excitations are very different from the single particle excitations in a noninteracting system. Nevertheless, we predict that the statistical distribution of level spacings is unchanged by mapping the non-Hermitian Bogoliubov operator to a real symmetric matrix. We numerically test our prediction by using a phase shift method for calculating the excitation energies.

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In recent years, the realization of Bose-Einstein condensation (BEC) of dilute gases [1] has opened new opportunities for studying dynamical systems in the presence of many-body interactions. However, most previous investigations have focused on one-dimensional or high-dimensional separable systems, and the dynamics of BEC in nonseparable systems with two or more degrees of freedom have not received much attention [2].

In the linear Schrödinger equation, systems with two or more degrees of freedom can be characterized by the statistics of energy levels: the typical distribution of the spacing of neighboring levels is Poisson or Gaussian ensembles for separable or nonseparable systems, respectively [3]. In the limit of short wavelengths (geometric optics) [4], classical trajectories emerge from the linear Schrödinger equation, and the two types of quantum statistics have been linked to different classical behaviors: Poisson to regular motion, while Gaussian ensembles to chaotic motion. It is natural to ask whether these findings for the linear Schrödinger equation can be generalized to other types of wave equations [5]. The Bogoliubov equation [6] obtained from the linearization about the ground state of the Gross-Pitavetskii (GP) equation has a purely real spectrum, and there is also a classical limit in the sense of geometric optics. It therefore makes sense and will be very interesting to explore the relationship between the Bogoliubov level statistics and regularity of the corresponding classical trajectories.

There is, however, an important difference between the two types of equations: while the Schrödinger equation is Hermitian, the Bogoliubov equation is non-Hermitian and its statistics cannot readily be predicted by standard random matrix theory. In fact, the Bogoliubov equation belongs to the category of symplectic problems, describing linearized motion about stationary states in nonlinear classical Hamiltonian systems. This can be easily understood by noting that the GP equation does have a classical Hamiltonian structure (of infinite dimensions, though) [7] and that the Bogoliubov equation describes excitations about a stationary solution of the GP equation. The non-

Hermiticity of the Bogoliubov equation makes it allowable to have complex eigenvalues in general, which signifies the instability of the stationary solution. This will not happen about the ground state (lowest energy state), which is always stable. Therefore, our investigation of the Bogoliubov problem should shed light on the behaviors of motions around stable stationary states in extensive classical Hamiltonian systems.

In this Letter, we investigate the level statistics of Bogoliubov excitation in separable circular as well as nonseparable stadium billiards. These are the excitations of a system of interacting particles in contrast with the modes of noninteracting particles described by the linear Schrödinger equation [8,9]. The classical trajectories of Bogoliubov waves are found to be regular (chaotic) in circular (stadium) billiards. By mapping the non-Hermitian Bogoliubov operator to a real symmetric matrix, we find the mean field interactions in the condensate do not change the level statistics of Bogoliubov excitations. This result is tested numerically by using a phase shift method for calculating the excitation energy. In the regime of strong interaction and low excitation energy (phonon), we map the Bogoliubov equation to an equivalent Schrödinger equation with a Neumann boundary condition and show that the statistics of Bogoliubov levels are the same as that for the Schrödinger equation, although interactions in the condensate do change the average number of levels up to a certain energy.

Consider condensed atoms confined in a quarter-stadium shaped trap of area  $A$  (with length of the top straight side  $L$ , radius of the semicircle  $R$ ) and height  $d$ , where  $d \ll R$  so that lateral motion is negligible and the system is essentially two dimensional [10]. With only a quarter of a stadium, one is restricted to a single symmetry class of the full problem [8]. The dynamics of the BEC are described by the GP equation

$$i \frac{\partial}{\partial t} \psi = -\frac{1}{2} \nabla^2 \psi + gN|\psi|^2 \psi, \quad (1)$$

where  $g = 2\sqrt{2}a/d$  is the scaled strength of nonlinear

interaction,  $N$  is the number of atoms, and  $a$  is the  $s$ -wave scattering length. The ground state of BEC can be written as  $\psi = \psi_0(\vec{r}) \exp(-i\mu t)$ , where  $\mu$  is the chemical potential and  $\psi_0(\vec{r})$  can be taken as real. The length and the energy are measured in units of  $b = \sqrt{4A/\pi}$  and  $\hbar^2/mb^2$ , respectively, so that the scaled area of billiards is  $\tilde{A} = \pi/4$ . The dynamics of the elementary excitations are obtained by linearizing the GP equation about the ground state  $\psi$  and their energy spectrum is described by the time-independent Bogoliubov equation [6]

$$\mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{L} = \sigma_z \begin{pmatrix} H_1 & H_2 \\ H_2 & H_1 \end{pmatrix}, \quad (2)$$

where  $\sigma_z$  is the Pauli matrix,  $H_1 = -\frac{1}{2}\nabla^2 + 2gN\psi_0^2 - \mu$ ,  $H_2 = gN\psi_0^2$ ,  $E$  is the excitation energy, and  $(u, v)$  is the eigenfunction of linear operator  $\mathcal{L}$ .  $\psi_0$   $[(u, v)]$  has the normalization  $\int_{\tilde{A}} \psi_0^2 dx dy = 1$  [ $\int_{\tilde{A}} (u^2 - v^2) dx dy = 1$ ] and satisfies the boundary condition  $\psi_{0\partial\tilde{A}} = 0$   $[(u, v)_{\partial\tilde{A}} = 0]$ .

Classical trajectories or rays arise from the Bogoliubov equation in geometric optics approximation [2,4]. Assume a trial Bogoliubov wave of the form  $(u, v) = (\alpha, \beta)e^{iS}$  and consider a slowly varying medium ( $\nabla\alpha$ ,  $\nabla\beta$  small) and a slowly varying velocity ( $\nabla^2 S$  small) approximation. We obtain the Eikonal equation  $|\nabla S|^2 = p^2$ , with  $p = \sqrt{\mu + \sqrt{E^2 + (gN\psi_0^2)^2} - 2gN\psi_0^2}$ . The classical trajectories of the Bogoliubov waves are still governed by the ray equation  $\frac{d}{dw}(p\hat{w}) = \nabla p$ , where  $\hat{w}$  is the direction of the trajectory and  $w$  is the arc-length coordinate along the trajectory. Interestingly, the classical trajectories are simply straight lines for noninteracting as well as interacting uniform gases because  $p = \sqrt{E_k}$ , the local momentum in both cases. In the regime of strong interaction where the ground state is nearly uniform, the classical trajectories of Bogoliubov waves are straight lines and undergo elastic specular reflection law at the boundary of the billiard. Therefore we predict that the Bogoliubov level statistics are still Poisson in circular billiards and Gaussian orthogonal ensembles (GOE) in stadium billiards through quantum classical correspondence.

This prediction is supported by a general argument based on mapping the non-Hermitian Bogoliubov operator  $\mathcal{L}$  to a real symmetric matrix. The linear operator  $\mathcal{L}$  can be written as  $\mathcal{L} = \sigma_z Q$ , where  $Q$  is a real symmetric positive definite matrix because the ground state of BEC is thermodynamical stable [6]. The positive definiteness of  $Q$  yields the decomposition  $Q = T^\dagger T$  ( $T$  is a real matrix with nonzero eigenvalues) and the Bogoliubov equation reduces to

$$T\sigma_z T^\dagger \begin{bmatrix} T(u) \\ T(v) \end{bmatrix} = E \begin{bmatrix} T(u) \\ T(v) \end{bmatrix}. \quad (3)$$

Therefore Bogoliubov excitation energy is the eigenvalue of a real symmetric matrix  $T\sigma_z T^\dagger$  and should have GOE

distribution in stadium billiards for arbitrary interaction strength [3].

In the following, we report a numerical test of this prediction by developing a phase shift method to calculate the Bogoliubov excitation energy. The condensate density is nearly uniform [11] in the interior of billiards that yields the plane wave forms of the Bogoliubov excitations. Similar to the scattering wave method in quantum mechanics [12], the nonuniform condensate density close to the boundary and the hard walls can be taken as a pseudopotential, and the scattering by this pseudopotential induces only a phase shift of the interior excited plane wave. The phase shift may be determined by solving the one-dimensional GP equation with an infinite wall at  $x \leq 0$ . Far from the wall, the Bogoliubov equation has both plane wave and exponential solutions for a certain excitation energy. We numerically integrate [13] the one-dimensional Bogoliubov equation with two different initial conditions to eliminate the exponential terms and to extract the plane wave solutions. The phase shift  $\delta$  is obtained by comparing the numerical solution with the expected  $\sin(kx + \delta)$  dependence. The result is shown in Fig. 1 as a function of  $k/\sqrt{g'}$ , where  $g' = gN\varphi_0^2$ , and  $\varphi_0$  is the condensate wave function far from the wall. We see that the phase shift approaches  $\pi/2$  in phonon regime and asymptotically approaches zero in free particle regime (high excitation energy and weak interaction).

With the phase shift method, we can calculate the Bogoliubov excitation energy in one-dimensional billiards. The reflected plane waves from the boundary walls  $x = 0$  and  $L$  can be written as  $\phi_0 = C \sin(kx + \delta_k)$  and  $\phi_L = F \sin[-k(x - L) + \delta_k]$ , respectively. The continuum of the wave function and its derivative in the interior of the billiards require  $\phi_0/\phi'_0 = \phi_L/\phi'_L$  that yields the quantization condition

$$kL + 2\delta_k = n\pi, \quad (4)$$

where  $n$  is an integer. Equation (4) determines the excitation energy  $E = \sqrt{(k^2/2)[(k^2/2) + 2g']}$  in one-dimensional billiards. For the noninteracting case ( $g = 0$ ), the phase shift  $\delta = 0$  and Eq. (4) reduces to  $k = n\pi/L$ , the quantization condition for a single particle. As  $k$  approaches zero ( $E \rightarrow 0$ ),  $\delta_k \rightarrow \pi/2$ ; therefore

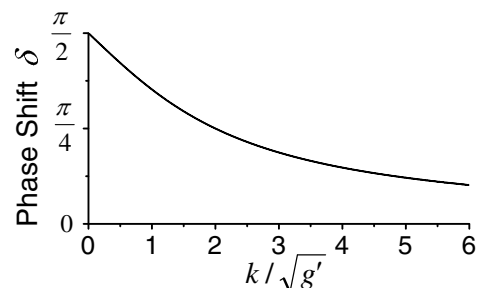


FIG. 1. Plot of phase shift  $\delta$  versus  $k/\sqrt{g'}$ .

TABLE I. Comparison of Bogoliubov excitation energies in one-dimensional billiards using both phase shift ( $E_1$ ) and matrix diagonalization methods ( $E_2$ ).  $gN = 1000$ .

$E_1$	$E_2$	$E_1$	$E_2$	$E_1$	$E_2$	$E_1$	$E_2$
0	2E-12	907.4	909.7	2144.4	2147.4	3894.3	3897.5
105.8	102.6	1038.7	1041.1	2333.0	2336.0	4153.1	4156.6
212.2	213.6	1175.9	1178.3	2529.7	2532.8	4421.4	4424.9
320.1	322.0	1319.6	1321.9	2735.3	2738.2	4699.1	4702.6
430.4	432.4	1470.0	1472.3	2949.0	2952.3	4968.1	4989.6
543.6	545.6	1627.0	1629.7	3171.9	3175.1	5282.2	5286.0
660.4	662.5	1791.7	1794.5	3403.6	3406.9	5588.4	5591.9
781.5	783.6	1964.1	1967.0	3644.1	3647.6	5903.7	5907.3

$k = 0$  ( $E = 0$ ) is a solution of Eq. (4) that corresponds to the ground state of BEC (uniform density indicates the  $\pi/2$  phase shift).

To check the validity of the phase shift method, we also calculate the excitation energies using a traditional matrix diagonalization method in which  $\mathcal{L}$  is represented as a matrix and the diagonalization process gives  $E$ . The results are compared with those from the phase shift method in Table I. We see that the phase shift method gives accurate results for the excitation energy. The wave function of the first three excited states is shown in Fig. 2. Clearly, the excitation wave function is described by the plane wave  $\sin(kx + \delta)$  in the interior of the billiards and drops to zero at the boundary.

The excitation spectrum in the free particle regime ( $k^2/2 \gg 2g'$ ) has been well understood [8,9]. Direct calculations of Bogoliubov excitation energies in two-dimensional billiards for arbitrary interaction strength and excitation energy are difficult for both phase shift and matrix diagonalization methods. However, we are more interested in the phonon regime ( $k^2/2 \ll 2g'$ ), where the effect of interactions is essential. In the phonon regime, the phase shift is approximately  $\pi/2$ , as seen from Fig. 1, which means that the first derivative of the plane wave should be zero at the boundary instead of the

zero wave function for the single particle. The Bogoliubov wave function far from the wall can be written as  $(u_k, v_k) = (U_k, V_k)\phi$ , where  $(U_k, V_k) = \frac{1}{2}(\chi + \chi^{-1}, \chi - \chi^{-1})$ ,  $\chi = [(k^2/2)/(k^2/2 + 2g')]^{1/4}$  [6],  $\phi$  satisfies the Schrödinger equation with Neumann boundary condition

$$\nabla^2 \phi + k^2 \phi = 0, \quad \partial \phi / \partial \vec{n} = 0, \quad (5)$$

and  $\vec{n}$  is the normal direction of the billiard wall. Equation (5) has an analytical solution  $\phi = BJ_{2m}(kr) \times \cos(2m\theta)$  in quarter-circular billiards, where  $J_{2m}(kr)$  is the Bessel function,  $m$  is an integer, and  $B$  is a normalization constant. The boundary condition is satisfied by requiring  $J'_{2m}(kR) = 0$ . No analytical solution is available for stadium billiards and we use the ansatz of the superposition of plane waves  $\phi(x, y) = \sum_{j=1}^M a_j \cos(k_{jx}x) \times \cos(k_{jy}y)$  (plane wave decomposition method [14]), where  $k_{jx} = k \cos(\theta_j)$ ,  $k_{jy} = k \sin(\theta_j)$ ,  $M$  is the number of plane waves, and  $\theta_j = 2j\pi/M$ . The ansatz solves Eq. (5) inside the billiards and the boundary condition determines the excitation energy.

The average number of energy levels  $\langle N(E) \rangle$  up to  $E$  should satisfy a Weyl-like type formula [15]

$$\langle N(E) \rangle = (\mathcal{A}E' + \mathcal{D}\sqrt{E'} + C)/4\pi, \quad (6)$$

where  $E' = k^2 = 2g'[\sqrt{1 + (E/g')^2} - 1]$ ,  $\mathcal{A}$  and  $\mathcal{D}$  are the area and the perimeter of the billiard, and  $C$  is a constant related to the geometry and topology of the billiard boundary. Equation (6) is valid only in the semiclassical limit  $E' = k^2 \gg (\frac{\mathcal{D}}{\mathcal{A}})^2$  that yields the Bogoliubov energy  $E/\sqrt{g'} \gg \frac{\mathcal{D}}{\mathcal{A}}\sqrt{1 + (1/4g')(\frac{\mathcal{D}}{\mathcal{A}})^2}$ . In the limit of large interaction constant ( $E/g' \ll 1$ ), we have  $\langle N(E) \rangle \approx \frac{1}{4\pi}(\mathcal{A}\{E^2/g'\} - [E^4/2(g')^3]) + \mathcal{D}\sqrt{E^2/g'} + C$  with the condition  $\frac{\mathcal{D}}{\mathcal{A}} \ll E/\sqrt{g'} \ll \sqrt{g'}$ .

We have computed the Bogoliubov excitation energies in circular billiards ( $R = 1, L = 0$ ) using the analytical solution and in stadium billiards [ $R = L = \sqrt{1/(1 + 4/\pi)}$ ] using plane wave decomposition method. In Fig. 3, we plot  $\langle N(E) \rangle$  in both circular and stadium billiards. The agreement between the numerical results and the corresponding Weyl-like formula [Eq. (6)] is

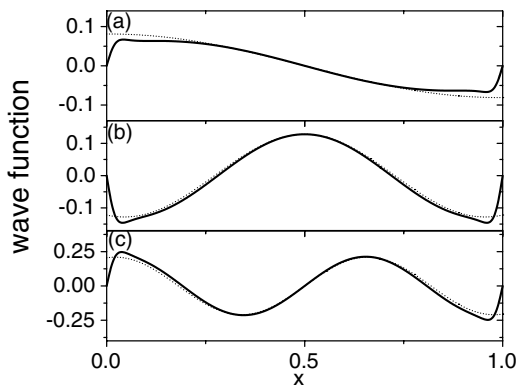


FIG. 2. Bogoliubov excitation wave functions in one-dimensional billiards for  $gN = 1000$ . Solid lines are from the matrix diagonalization method and dotted lines are from phase shift method  $\sin(kx + \delta)$ . (a), (b), and (c) represent the first three excitation wave functions, respectively.

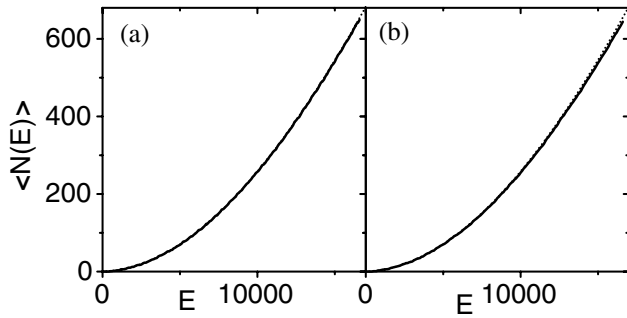


FIG. 3. Plots of the average number of energy levels  $\langle N(E) \rangle$  up to energy  $E$  in quarter-circular (a) and quarter-stadium (b) billiards.  $g' = gN\varphi_0^2 = 25000$ . Solid and dotted lines are from numerical results and Eq. (6), respectively.

satisfactory. We unfold the spectrum formed by the energies  $E_n$ ; i.e., we evaluate Eq. (6) for each  $E_n$  in order to obtain the new energies  $\tilde{E}_n = \langle N(E_n) \rangle$ . Note that the integer part of  $\tilde{E}_n$  is about  $n$  and, as a result, the corresponding mean level spacing is characterized through  $\langle s \rangle = \sum (\tilde{E}_{n+1} - \tilde{E}_n) / \langle N \rangle \approx 1$ . The resulting level spacing distributions  $P(s)$  are shown in Fig. 4. Clearly, the statistics of the Bogoliubov excitation energy level spacing are still Poisson in circular billiards and GOE in stadium billiards.

Experimentally, the system can be realized by confining BEC in two-dimensional optical billiards [16]. Atoms could be trapped in a one-dimensional optical lattice that is in the vertical direction to counteract gravity. Transverse motion could be confined to a closed pattern of light that is tuned to the blue side of atomic resonance, forming a repulsive barrier for the atoms. This pattern can be created by rapid scanning of a beam as was

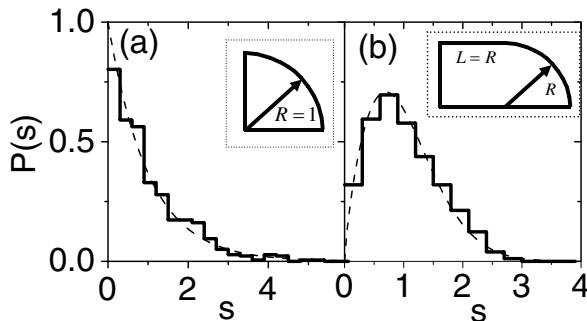


FIG. 4. The distributions of the spacing of neighboring Bogoliubov energy levels ( $k < 100$  and the lower 50 ones are omitted) in phonon regime. (a) Circular billiards. Dashed line: Poisson distribution  $P_0(s) = \exp(-s)$ . (b) Stadium billiards. Dashed line: Brody distribution  $P(s) = (\nu + 1)a_\nu s^\nu \exp(-a_\nu s^{\nu+1})$  from the prediction of GOE, where  $a_\nu = [\Gamma(\frac{\nu+2}{\nu+1})]^{\nu+1}$  and  $\Gamma$  is Gamma function. The fitting Brody parameter  $\nu = 0.76$  is smaller than expected (0.953) due to the “bouncing ball states” [9] and finite levels effects [19].

demonstrated for ultracold atoms. However, for the case of a BEC it would be better to create static patterns, which can be accomplished using a liquid crystal spatial light modulator [17]. The interaction strength may be adjusted by the confinement in the vertical direction or by the number of atoms in each node of the standing wave. The Bogoliubov excitation energy may be measured using Raman transition between two hyperfine ground states, denoted  $|1\rangle$  and  $|2\rangle$ , respectively. The condensate would be formed in state  $|1\rangle$ . Two copropagating Raman beams would drive a transition to state  $|2\rangle$ , and the number of atoms in that state would be measured as a function of the frequency difference in the beams. The coupling efficiency to the excited states must depend upon the spatial profiles of the beams, requiring more detailed analysis [18].

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