OBSERVATION OF A STRANGE ATTRACTOR

J.-C. ROUX,* Reuben H. SIMOYI and Harry L. SWINNEY

Department of Physics, The University of Texas, Austin, Texas 78712, USA

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Phase space portraits have been constructed and analyzed for noisy (nonperiodic) data obtained in an experiment on a nonequilibrium homogeneous chemical reaction. The phase space trajectories define a limit set that is an "attractor" - following a perturbation, the trajectory quickly returns to the attracting set. This attracting set is shown to be "strange" - nearby trajectories separate exponentially on the average. Moreover, the Poincaré sections exhibit the stretching and folding that is characteristic of strange attractors.

1. Introduction

Recent observations of nonperiodic behavior in nonlinear chemical, hydrodynamic, and other systems have been described as being characterized by strange attractors [1, 2]. While the data have been shown to exhibit some features of deterministic chaos, the evidence supporting the use of the term "strange attractor" has been incomplete. In this paper we will examine carefully a set of nonperiodic data and describe the construction and analyses of phase portraits that strongly support the strange attractor appellation for these data. Terms such as phase portrait and strange attractor [3–5] will be defined in a simple intuitive manner; readers familiar with these concepts may simply examine the figures to see how the concepts have been implemented with laboratory data.

The data analyzed here were obtained in our experiments on the Belousov–Zhabotinskii reaction in a well-stirred flow reactor [2]. In this reaction, which involves more than 30 chemical constituents, an acidic bromate solution oxidizes malonic acid in the presence of a metal ion catalyst. The concentration of one of the chemicals, the bromide ion, was measured with a specific ion probe, digitized, and recorded as a function of time in a computer in files of 32768 points.

Periodic and nonperiodic states can be distinguished by their power spectra, as fig. 1 illustrates. The spectrum for a periodic state consists of a sharp fundamental frequency component and its harmonics, while the spectrum for a nonperiodic state contains broadened spectral lines or, as in fig. 1, broadband noise. The analysis presented in the following sections indicates that the broadband noise in the spectrum for the nonperiodic state in fig. 1 is primarily deterministic rather than stochastic (e.g., environmental or thermal) in origin.

We will describe phase portraits in section 2, Poincaré sections and maps in section 3, attractors in section 4, the determination of the largest Lyapunov exponent in section 5, and stretching and folding of the attractor in section 6.

*Permanent address: Centre de Recherche Paul Pascal, Université de Bordeaux-I, Domaine Universitaire, 33405 Talence Cedex, France.
2. Construction of phase portraits

Around the turn of the century Poincaré and others recognized that much could be learned about dynamical behavior from an analysis of system trajectories in a multi-dimensional phase space in which a single point characterizes the entire system at an instant of time. The set of phase space trajectories for all possible initial conditions (for a given set of control parameter values) forms a phase portrait of the system.

The $N$-dimensional phase portrait describing the well-stirred (that is, homogeneous) Belousov–Zhabotinskii system could be constructed from measurements of the time dependence of the concentration of all $N(=30+)$ chemical species in the reaction. Fortunately, such a difficult task is unnecessary – a multi-dimensional phase portrait can be constructed from measurements of a single variable by a procedure proposed by Ruelle [6] and Packard et al. [7]. The idea, which is justified by embedding theorems [8, 9], is as follows: For almost every observable $B(t)$ and time delay $T$ an $m$-dimensional portrait constructed from the vectors $\{B(t_0), B(t_0 + T), \ldots, B(t_0 + (m-1)T)\}$, where $t_k = k\Delta t$, $k = 1, 2, \ldots, \infty$, will have the same properties (for example, the same spectrum of Lyapunov exponents) as one constructed from measurements of $N$ independent variables, if $m \geq 2N + 1$. Strictly speaking, the phase portrait obtained by this procedure gives an embedding of the original manifold [8, 9].

The choice of the time delay $T$ is almost but not completely arbitrary. An obvious example of the necessity of the phrase “for almost any $T$” would be a time delay equal to the period of a periodic
attractor is viewed as it is continuously rotated about an axis [10]. Thus even for these chaotic data the character of the phase portrait is clear in projections of far fewer dimensions than the 61 + required to satisfy the embedding theorems; this low dimensionality of the phase portrait is revealed even more clearly in the Poincaré section.

3. Poincaré section and map

Rather than analyze phase portraits directly it is easier to analyze the lower-dimensional Poincaré section which is formed by the intersection of “positively directed” orbits of an $m$-dimensional phase portrait with an $(m - 1)$-dimensional hypersurface. A 2D Poincaré section constructed for a 3D phase portrait of a nonperiodic state is shown on the left-hand side of fig. 5. (The right-hand side of the figure shows the effect of perturbations, as discussed in the following section.)

The points on the Poincaré section [fig. 5(b)] lie to a good approximation along a parameterizable curve, nor on a higher dimensional curve. (However, the actual dimension of the Poincaré section must be at least slightly greater than unity because of the fractal nature of attractor [11, 12].) Therefore, the coordinate values at successive intersections provide a sequence $\{X_n\}$ which defines a 1D map, $X_{n+1} = f(X_n)$, as shown in fig. 5(c). The data appear to fall on a single-valued curve, indicating that the system is deterministic – for any $X_n$, the map determines $X_{n+1}$.

4. An attractor: response to perturbations

The right-hand side of fig. 5 illustrates that the post-transient set described by the phase space trajectories is really an attractor: the trajectories rapidly return to this limit set after finite perturbations. The basin of attraction of an attractor is the set of all initial conditions for which the trajectories asymptotically approach the attractor. For the state illustrated in fig. 5 the trajectories...
have been found to return to the attractor for all perturbations we have used, so this attractor could be globally attracting. Perturbations have included injections of bubbles into one of the feed lines, injections of bromide ions into the reactor, and turning off the stirrer or one of the chemical feed lines for a few seconds. For other values of the control parameters multiple stable states have been observed [13], each with its own basin of attraction — the trajectories will then still return to the original attractor for perturbations that are not too large, but a sufficiently large perturbation can send the system trajectory from one basin of attraction into another basin.

5. A Lyapunov exponent

A quantitative measure of nonperiodic (chaotic) behavior is provided by the value of the largest Lyapunov exponent, which characterizes the average rate of separation of nearby trajectories [14]. This exponent is positive for a chaotic state, while for a periodic state it is zero. An attractor with a
Fig. 4. The 3D character of the phase portrait for the nonperiodic state is illustrated by these 2D projections (of the 3D phase portrait) rotated about the ordinate in angular steps of $9^\circ$. The 3D portrait is given by $[B(t), B(t + T), B(t + 2T)]$ with $T = 52.8$ s (about one-half the average time per orbit).
Fig. 5. A chaotic state in the absence of an external perturbation (left-hand side) and with a perturbation (right-hand side): (a) A 2D projection of a 3D phase portrait. (b) A Poincaré section constructed by the intersection of positively directed trajectories with the plane (normal to the paper) passing through the dashed line in (a). (c) A 1D map constructed by plotting as ordered pairs \((X_{n-1}, X_n)\) the successive values of the ordinate of trajectories when they cross the dashed line in (a).
positive Lyapunov exponent exhibits sensitive dependence on initial conditions – trajectories starting from two nearby points will evolve quite differently in time [15]. Thus all information about the initial conditions is rapidly lost, since any uncertainty in the initial values will be magnified until it becomes as large as the attractor.

The largest Lyapunov exponent for the data discussed here can be computed from the 1D map: the Lyapunov exponent for a set of data \( \{X_i\} \), \( i = 1, \ldots, n \), described by a 1D map is given by [14]

\[
\lambda = \frac{1}{n} \sum_{i=1}^{n} \ln|f'(X_i)|,
\]

where \( f'(X) \) is the derivative of the map at \( X \). We have calculated \( f'(X) \) at each point by fitting the data with cubic splines [16]. Fig. 6(a) shows a spline fit to a set of data \( \{X_i\} \) and fig. 6(b) shows \( \ln|f'(X)| \) computed from the fit. The resultant Lyapunov exponent value for these data is \( 0.3 \pm 0.1 \).

We have determined \( \lambda \) for several different sets of data and the results have been found to be alarmingly sensitive to the scatter in the data and the number and placement of the knots in the spline fit. A drift or scatter of several percent in the data defining a map with 100 to 200 points yielded in some cases values of \( \lambda \) ranging from positive to negative, depending on the fitting procedure. The sensitivity to noise and to the data fitting procedure, to be discussed in detail elsewhere [17], has not been adequately emphasized in past reports of the determination of \( \lambda \) [2, 18, 19]. On the other hand, for the data in fig. 6 the value of \( \lambda \) was reasonably robust under changes in the number and positions of the knots in the spline fit and under changes in the number of data points in the fit. From this analysis it is concluded that \( \lambda = 0.3 \pm 0.1 \) for these data.

The experimentally determined value of \( \lambda \) is positive, yet well below the maximum value for a 1D map, \( \lambda_{\text{max}} = \ln 2 = 0.69 \) [4]. Comparison of an experimental value of \( \lambda \) with theory is difficult since the dependence of \( \lambda \) on bifurcation parameter is extremely complex (see figs. 3 and 4 of [20]). However, the presence of a small amount of external noise, invariably present in any experiment, tends to smooth this function to a simple power law behavior (see fig. 10 in [20]). We have examined the effect of the external noise on our data by

![Fig. 7. A schematic representation of the stretching and folding of the X-axis described by the 1D map.](image-url)
fitting the data to the function
\[ X_{n+1} = aX_n \exp(-bX_n), \]  
(2)
which describes the observed maps rather well [17]. For the data in fig. 6, which correspond to a chaotic state near the periodic 3-cycle of the U-sequence [21], the qualitative form of the scatter in the data and the observed value of \( \lambda \) are both reproduced by adding the same amount of multiplicative noise to eq. (2). Finally, the value 0.3 ± 0.1 may also be compared with the topological entropy of a 3-cycle, \( \ln[(1 + \sqrt{5})/2] = 0.48 \) [22], which is an upper bound for the Lyapunov exponent.

6. A strange attractor (stretching and folding)

A strange attractor is an attractor with at least one positive Lyapunov exponent. Therefore, figs. 3–5 show strange attractors.

The exponential divergence of nearby trajectories implies that there must be a stretching of the attractor as trajectories evolve, but since the attractor lies within a bounded region of phase space, the attractor must also exhibit folding. Indeed, stretching and folding is a hallmark of strange attractors.

The stretching and folding can be seen in the 1D map, as fig. 7 illustrates. The stretching of the attracting sheet itself is illustrated schematically in

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Fig. 8. The stretching of a segment of length \( d \) in the Poincaré section in (a) is illustrated schematically in (b). Trajectories in the sheet-like attractor separate exponentially, while trajectories off of the attractor converge exponentially to it.

Fig. 9. A 2D projection of a 3D phase portrait, indicating the locations of the Poincaré sections shown in fig. 10. The Poincaré sections are planes normal to the paper, passing through the dashed lines.
Fig. 10. Successive Poincaré sections illustrating the stretching and folding that is characteristic of strange attractors. For the convenience the sections are drawn equal in size; the actual relative sizes (height × width) are: (1) 0.3 × 0.2, (2) 1.1 × 0.9, (3) 1 × 1, (4) 0.5 × 0.9, (5) 0.5 × 0.9, (6) 0.5 × 0.8, (7) 0.4 × 0.5, (8) 0.4 × 0.2, (9) 0.4 × 0.08.

Fig. 8 [23]. Figs 9 and 10 demonstrate that the stretching and folding can be directly observed by analyzing the evolution of Poincaré sections at successive positions along the attractor.

In summary, we have shown that a nonperiodic state observed in an experiment on the Belousov–Zhabotinskii reaction is described by a phase portrait that has the properties of a strange attractor.

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References


[3] For a discussion of these terms see, for example, J.P. Eckmann, “Roads to turbulence in dissipative dynamical systems” Rev. Mod. Phys. 53 (1981) 643; see also [4] and [5].


[16] IML Cube splines fit, Routine DCSEU (IMSL, Sixth Floor, GNB Building, 7500 Bellaire Boulevard, Houston, Texas 77036 USA).


