Virtual Seafloor Reduces Internal Wave Generation by Tidal Flow

Likun Zhang* and Harry L. Swinney†
Department of Physics and Center for Nonlinear Dynamics, University of Texas at Austin, Austin, Texas 78712, USA
(Received 19 August 2013; published 11 March 2014)

Our numerical simulations of tidal flow of a stratified fluid over periodic knife-edge ridges and random topography reveal that the time-averaged tidal energy converted into internal gravity wave radiation arises only from the section of a ridge above a virtual seafloor. The average radiated power is approximated by the power predicted by linear theory if the height of the ridge is measured relative to the virtual floor. The concept of a virtual floor can extend the applicability of linear theory to global predictions of the conversion of tidal energy into internal wave energy in the oceans.

DOI: 10.1103/PhysRevLett.112.104502 PACS numbers: 47.35.Bb, 47.55.Hd, 92.10.hj, 92.10.Hm

The oceans are stratified as a consequence of decreasing temperature and increasing salinity with depth. Buoyancy provides a restoring force for density perturbations that are produced, for example, by tidal flow over bottom topography [1–3]. Tidal flow over topography generates internal gravity waves, called internal tides [4,5], which play a role in ocean mixing and circulation [6–8].

An understanding of the energy budget of the oceans requires a determination of the efficiency of conversion of tidal energy into internal gravity wave energy [9–14]. The conversion rate for topography of small slope and small height $H$ in a deep ocean (weak topography) is given by linear theory, which predicts the main topographic dependence as $H^2$ [15]. This $H^2$ dependence has also been found to describe steep isolated ridges in the deep ocean, except for a prefactor that is about twice as large for small slopes [16–20]. In contrast, recent studies of periodic steep topography suggest that wave interference between neighboring ridges can suppress the $H^2$ dependence [21–23]. Complex topographies can also suppress tidal conversion [24] and generate wave attractors [25]. Thus, considerations of weak or isolated topography are not sufficient for predicting tidal conversion in the deep ocean [26].

Here we examine in numerical simulations how wave interference suppresses tidal conversion for periodic knife-edge and random topography, and we explore an extension of linear theory to such topographies. The simulations are conducted for tidal flow of a uniformly stratified fluid with constant buoyancy frequency, $N = \sqrt{-(g/\rho_0)(dp/dz)}$ [with $\rho(z)$ the vertical density distribution, $g$, the gravitational acceleration, $\rho_0$, a reference density]. The internal waves radiated by the topography have a beam slope (relative to the horizontal direction) [1–3],

$$S_{IW} = \sqrt{\frac{1 - (f/\omega)^2}{(N/\omega)^2 - 1}},$$

where $\omega$ is the tidal frequency and $f$ is the Coriolis parameter. We vary $S_{IW}$ for fixed topography and examine the wave drag and the time-averaged radiated energy flux.

We find that for both periodic and random topographies, the time-averaged internal wave power is generated only above an elevated virtual seafloor between neighboring ridges, and this reduces the tidal conversion.

Methods.—We conduct two-dimensional direct numerical simulations of the Navier-Stokes equations in the Boussinesq approximation using the CDP-2.4 code (subgrid modeling is disabled) [27], modified to include buoyancy effects [28] and validated in studies of internal waves [28–31]. Tidal flow is produced by adding to the momentum equation a horizontal body force, $F(t) = \rho_0 U_0 \sin(\omega t)$, where $\rho_0 = 10^3$ kg/m$^3$, $U_0 = 0.14$ cm/s, and $\omega = 1.4 \times 10^{-4}$ rad/s (the $M_2$ tidal frequency). The resultant barotropic tide closely approximates a horizontal oscillating flow $U(t) = U_0 \sin(\omega t)$ with tidal excursion $A = U_0/\omega = 10$ m, sufficiently small to avoid overturning and turbulence. We neglect rotation, i.e., $f = 0$.

A periodic array of knife edges is modeled by narrow top hats of height $H = 100A$ and width $W = 1.6A$, separated by $L = 200A$; some computations are also made with $(W, H, L)/A = (1.6, 50, 150)$ and $(1.6, 50, 200)$. Also, random topography is generated from an ensemble of Gaussian random processes with a spectrum, $S(n) = 4H_0A \sqrt{4^2 + n^2}^{-5/2}$ (with mode number $n \leq 32$) [16], as suggested by statistical modeling of the seafloor [15,32]. We choose $L = 100A$ and height $H_0 = 33A$, which yields a topographic peak-to-peak height $H = 100A$.

The beam slope $S_{IW}$ is changed over a wide range by varying stratification $N$, which leads to a corresponding variation of the Froude number, $Fr = U_0/NH \sim 10^{-2}–10^{-1}$. The fluid is isothermal with a large viscosity, $\nu = 10^{-2}$ m$^2$/s, to save computational expense, but the viscosity has a negligible effect on the tidal conversion in laminar flow [31]. The Reynolds number is $Re = HU_0/\nu \sim 100$. The salt diffusivity, $\kappa = 2 \times 10^{-5}$ m$^2$/s, has a negligible effect for the duration of our simulations because of the large Schmidt number, $\nu/\kappa = 500$.

Our computations are for a domain of width $-L/2 < x < L/2$ and height $0 < z < H + 1450A$ with periodic boundary conditions on the sides. The topography is

$\frac{\partial \rho}{\partial t} + \frac{\partial \rho (\mathbf{U} \cdot \nabla \mathbf{U})}{\partial x} = -\nabla \cdot (\rho \mathbf{F}) + \frac{1}{\rho} \nabla \cdot \mathbf{F} \nabla \cdot \mathbf{U}$
located at the bottom with a no-slip bottom boundary condition; the knife edge is centered at \( x = 0 \). To absorb upward-propagating internal waves, a sponge layer above \( z = H + 450 \text{A} \) is made by applying a Rayleigh damping force (proportional to the velocity deviation from the tidal velocity) that smoothly increases upward to the top boundary. The knife-edge (random topography) domain has a structured grid of \( 1 \times 10^6 \times (3 \times 10^6) \) control volumes. The horizontal resolution for the knife-edge domain varies smoothly from \( \Delta x = 0.08 \text{A} \) at the center to \( \Delta x = 0.8 \text{A} \) at the sides; for random topography the resolution is uniform, \( \Delta x = 0.625 \text{A} \). The vertical resolution is \( \Delta z = 0.2 \text{A} \) at the bottom \( (z \leq H + 50 \text{A}) \) and gradually stretches to \( \Delta z = 2 \text{A} \) at the top boundary. To achieve a steady state, each simulation is run for 20 tidal periods with 2000 time steps per period. Numerical convergence is checked by doubling (halving) both spatial and temporal resolution, which leads to computed wave power changes smaller than 1% (3%).

**Virtual seafloor.**—The radiated power per ridge computed for a periodic array of knife edges is shown in Fig. 1 as a function of the ratio of the valley slope, \( S_{\text{valley}} = 2H/L \), to the wave beam slope. The power is computed from the time-averaged work rate [33], \( P = \langle D(t)U(t) \rangle \), where \( D(t) = \int_0^H \Delta p'(z,t) dz \) is the wave drag per ridge, \( \Delta p'(z,t) \) is the drop of the wave pressure \( p' \) across the knife edge from \( x = -W/2 \) to \( x = W/2 \), and the brackets denote the average over a tidal cycle. For internal wave beams \( (S_{\text{valley}}/S_{\text{IW}} \ll 1) \), the power per ridge \( P \) approaches that radiated by a single isolated ridge in the deep ocean, \( P_{\text{isolated}} = \frac{1}{2} \pi \rho_0 U_0^2 H^2 \sqrt{N^2 - \alpha^2} \) [17]. For shallow internal waves, constructive and destructive interference leads to oscillations in the power ratio with maxima and minima at \( S_{\text{valley}}/S_{\text{IW}} = 2n - 1 \) and \( 2n \), respectively (where \( n \) is an integer) [22]. The successive maxima, rather than being infinite as in [22], are finite and decay with increasing slope ratio (see also [23]).

We find the time-averaged power in the successive maxima is generated only above an elevated virtual seafloor (cf. dot-dashed lines in Fig. 1). To identify the virtual floor associated with the conversion of tidal energy into wave energy, we compute the power input by the tides over the part of a ridge below a height \( z \) from the wave drag, \( P_{\text{in}}(z) = \int D(z,t)U(t) \), where \( D(z,t) = \int_0^H \Delta p'(z',t) dz' \). For \( S_{\text{valley}}/S_{\text{IW}} = 1 \), the power \( P_{\text{in}}(z) \) increases from zero at the ocean bottom \((z = 0)\) to the total power at the top of the ridge (Fig. 2, top panel in the center). For the two lower panels, the power drops to zero at the virtual floor \( z = z_0 \), like that at \( z = 0 \), while above the virtual floor, the power increases with the height until reaching the total power at \( z = H \).

The converted tidal energy is radiated by internal waves with a time-averaged energy flux \( \Phi = \langle p'u' \rangle \) (Fig. 2, left-hand panels), where the wave velocity \( u'(x,z,t) \) is the difference between the fluid velocity \( u(x,z,t) \) and the barotropic tidal velocity \( u_{\text{baro}}(x,z,t) \) [33]. The radiated wave power crossing a horizontal plane, \( P_{\text{out}}(z) = \int_{L/2}^{L/2} \Phi_z(x,z) dx \), as shown by the red curves (center panels), is nearly equal to the input power \( P_{\text{in}}(z) \) for \( z \leq H \): the small difference arises from local dissipation. The upward radiation starts from the virtual floor, as no time-averaged energy crosses this floor.

The tidal conversion depends on the phase \( \phi(z) \) of the pressure drop \( \Delta p' \) relative to the tidal flow \( U(t) \). The phase factor \( \cos \phi(z) \) is shown in the right-hand panels of Fig. 2. For the first maximum (top), \( \cos \phi(z) > 0 \) along the whole ridge surface; thus, the entire ridge surface acts as a source with the time-averaged energy flux radiating away from the ridge (left panels). For higher maxima (two lower panels), \( \cos \phi(z) \) is positive above the virtual floor, while below the virtual floor \( \cos \phi(z) \) is alternately positive or negative along the surface; correspondingly, the time-averaged energy flux effectively radiates away from or towards the ridge (left panels).

The successive panels from top to bottom in Fig. 2 would be equivalent if the part below the virtual floors was deleted and the height of the ridge above the virtual floor, \( (H - z_0) = H/(2n - 1) \), was scaled to the full height \( H \). Thus, the power in successive maxima is \( P/P_{\text{isolated}} \propto (H - z_0)/H^2 = 1/(2n - 1)^2 \), as confirmed by the normalization \((2n - 1)^2\) of the flux and power ratio in Fig. 2 (except for small differences due to dissipation and boundary layers).

A virtual floor, where the tidal conversion below that floor is zero, exists for general values of \( S_{\text{valley}}/S_{\text{IW}} \), not only for the power maxima. This virtual floor is close to the level where downward beams generated by neighboring ridges first intersect (for \( S_{\text{valley}}/S_{\text{IW}} \geq 1 \); see sketch in Fig. 3),

\[
\frac{z_0}{H} = 1 - \left( \frac{S_{\text{valley}}}{S_{\text{IW}}} \right)^{-1},
\]

(Fig. 1 (color online). The radiated internal wave power per ridge for a periodic array of knife edges exhibits peaks (blue diamonds) due to constructive interference, but the successive peaks are suppressed as a consequence of an elevated virtual seafloor (horizontal dot-dashed lines); the radiated power originates only above this floor. The peaks are located at slope ratios \( S_{\text{valley}}/S_{\text{IW}} = 1, 3, 5 \), where \( S_{\text{IW}} \) is the slope of the wave beams (red lines). The peaks have relative values \( 1:1/3^2:1/5^2 \) when the radiated power is normalized by the power radiated by an isolated knife edge. The dashed lines in the inset illustrate the valley slope \( S_{\text{valley}} \), which is unity for the data plotted.)
while for $S_{\text{valley}}/S_{\text{IW}} \leq 1$, $z_0/H = 0$. As a result, the power $P$ oscillates about the power $P_{\text{isolated}}$ radiated by a single isolated ridge with height $(H - z_0)$; see Fig. 3. The three sets of data in Fig. 3, corresponding to different Re and Fr, agree except at the maxima there is a few percent difference associated with nonlinearity. Further, the average of $P/P_{\text{isolated}}$ between successive maxima is about 1/2 [34], in accord with the result from linear theory for a seafloor shifted from the real one at $z = 0$ to the virtual one at $z = z_0$.

**Virtual floor model and random topography.**—The concept of a virtual floor can extend the applicability of linear theory for weak topography to arbitrary topography. Internal wave beams can be constructed to obtain a virtual floor between each pair of adjacent topographic ridges. Then linear theory can be applied to the topography above the virtual floor to predict the conversion rate. The linear theory for weak topography gives the total wave power converted by tidal flow of an uniformly stratified fluid in the deep ocean [15],

$$P_{\text{linear}} = \left(\rho_0 U_0^2 \omega\right) \frac{\left[(N/\omega)^2 - 1\right]^{1/2}}{\left[1 - (f/\omega)^2\right]^{1/2}} g_{\text{topo}}.$$  \hspace{1cm} (3)

where the topographic dependence $g_{\text{topo}} = \left(L/2\pi\right) \times \int_{-\infty}^{\infty} k \mid \hat{h}(k)\mid^2 dk$ is computed from the Fourier transform $\hat{h}(k) = \int_{-\infty}^{\infty} \exp(-i k x) z(x) dx$ of the topography $z(x)$ with extent $L$ in the $x$ direction and unit length in the $y$ direction.

We illustrate the virtual floor model using sinusoidal topography, which serves as a Fourier basis for an arbitrary seafloor. For a one-dimensional sinusoidal basis $z(x) = \frac{1}{2} H \cos(kx + \phi)$, the power radiated for each topographic period ($L = 2\pi/k$) with small $kH$ is given by linear theory Eq. (3), $P_{\text{linear}} \propto H^2$. Our virtual seafloor model simply replaces the topographic height $H$ with the height $(H - z_0)$ measured relative to the virtual floor $z_0$ [cf. Fig. 4(a) inset]. In other words, we multiply the energy flux from linear theory by a weighting function, $(1 - z_0/H)^2$. The virtual floor height is $z_0/H = 0$ for $S_{\text{valley}}/S_{\text{IW}} \leq 1$, where the valley slope, $S_{\text{valley}} = (0.3623) kH$, is computed from the geometrical definition, $S_{\text{valley}} = \max \left[\frac{(z(x) - z(x_m))}{(x - x_m)}\right]$ with $z(x_m) = \min \left[z(x)\right]$. For $S_{\text{valley}}/S_{\text{IW}} > 1$, the virtual floor height,

$$z_0/H = 1 - \left[\epsilon - \sqrt{\epsilon^2 - 1 + \pi - \sin^{-1}(\epsilon^{-1})}\right]/2\epsilon.$$  \hspace{1cm} (4)

where $\epsilon = (1.38) S_{\text{valley}}/S_{\text{IW}} = (1) kH/S_{\text{IW}}$ follows from a determination of the point where the wave beam is radiated.

To test the proposed use of the virtual floor concept in linear theory, we compare in Fig. 4(a) our result for sinusoidal topography, $P = P_{\text{linear}}(1 - z_0/H)^2$, with an exact Green’s function analysis (magenta curve) [23]. The two results agree on the transition at $S_{\text{valley}}/S_{\text{IW}} = 1$, and
FIG. 4 (color online). Application of the virtual floor concept to sinusoidal and random topography. (a) For sinusoidal topography of height $H$ and virtual floor height $z_0$ (cf. inset), the internal wave power predicted using the virtual floor concept (black curve), $P = P_{\text{linear}}(1 - z_0/H)^2$, agrees well with the power predicted by a Green’s function approach (magenta curve, cf. Fig. 2 in [23]), if the comparison for $S_{\text{valley}}/S_{\text{TW}} > 1$ is made by averaging between adjacent maxima in the Green’s function result. The inset illustrates the valley slope $S_{\text{valley}}$ (slope of dashed line through the lowest point) and the internal wave beams (red lines) radiated from the points (red dots) where the beam slopes equal the mountain slopes. (b)-(c) For random topography the virtual floor is illustrated by (b) the vertical energy flux $\Phi_z$, where no time-averaged wave power crosses the virtual floor between adjacent ridges (horizontal dot-dashed lines), and by (c), where the total power radiated upward $P_{\text{out}}(z)$ starts from virtual floor (dot-dashed line) and increases up to the total power at $z = H$ (for larger $z$ the power decreases due to dissipation). Both the power and the flux are normalized by predictions of linear theory (where $\Phi_{\text{linear}} \equiv P_{\text{linear}}/L$).

both approach the linear theory prediction for $S_{\text{valley}}/S_{\text{TW}} \ll 1$ (small $kH$). For $S_{\text{valley}}/S_{\text{TW}} > 1$ (large $kH$) the prediction from the virtual floor approach agrees (within 5%) with the Green’s function result for the average power between successive maxima. The virtual floor prediction yields

$$P = (\alpha_0 U_0^2 \omega)(N/\omega)^2 - 1^{-1/2}[1 - (f/\omega)^2]^{1/2} k^{-2},$$

which recovers the independence of $H$ in [22,23]. In Eq. (5) we have used $(1 - z_0/H)^2 = a(kH/S_{\text{TW}})^{-2}$, where $a \in [3, \pi^2/8]$ follows from Eq. (4) [for periodic knife edges, $a = \pi^2/8$ follows from Eq. (2)]. The suppression in radiated power for $S_{\text{valley}}/S_{\text{TW}} \gg 1$ is far more significant than the increase around $S_{\text{topo}}/S_{\text{TW}} = 1$ (corresponding to $S_{\text{valley}}/S_{\text{TW}} = 0.7246$ in Fig. 4(a) with the topographic slope $S_{\text{topo}} \equiv \max|\partial z(x)/\partial x|$ [16]).

The time-averaged radiated internal wave power is suppressed as a consequence of virtual floors between adjacent topographic peaks, as Fig. 4(b) illustrates for internal waves generated by random topography that covers a wide range of slope ratio ($S_{\text{valley}}/S_{\text{TW}} \in [0, 1.94]$, $S_{\text{topo}}/S_{\text{TW}} \in [0, 3.43]$, and $S_{\text{TW}} = 1$). For this small horizontal scale and steep topography, the total radiated power, $P_{\text{out}}(z = H)$ in Fig. 4(c), is 55% of the power predicted by linear theory, $P_{\text{linear}}$ in Eq. (3). An average over a large number of realizations of random topography would provide an accurate determination of power suppression by the virtual floor.

Discussion.—Our examination of wave drag and energy flux for periodic knife-edge topography reveals that the tidal conversion into internal wave radiation arises only from topography that lies above an elevated virtual floor. The average radiated power is found to be given by linear theory for weak topography with the actual floor replaced by a virtual floor. The virtual floor picture is validated for a sinusoidal basis by comparison with a Green’s function analysis. The virtual floor is further shown to be applicable to random topography.

This study provides insight into the wave interference that suppresses the conversion of tidal energy for small horizontal scales and steep topography. The elevated virtual floor height $z_0$ for topographic valley slope greater than the internal wave slope leads to four consequences for the radiated wave power: (i) The power is approximately independent of $H$ because the height $(H - z_0)$ measured relative to the virtual floor is almost independent of $H$. (ii) To leading order the power has a $k^{-2}$ dependence that represents the reduction of the effective ridge height $(H - z_0)^2$ when the ridge separation $(2\pi/k)$ narrows. (iii) The power decreases with increasing stratification $N$ because the virtual floor is elevated when the beam slope becomes shallower [cf. Eq. (1)]. In contrast, linear theory predicts the power increases with increasing $N$ [Eq. (3)]. (iv) Rotation suppresses the power relative to linear theory because increasing the Coriolis parameter $f$ reduces the beam slope [Eq. (1)] and, hence, elevates the virtual floor.

The virtual floor concept can extend the applicability of linear theory to global random topography. High-resolution data for ocean topography [35] and stratification [36] could be used to map the seafloor to a virtual floor, and then linear theory could be applied to the virtual floor. The virtual floor for small scales, say less than 1 km where seafloor topographic data are scarce [32], could be included in the linear theory analysis by multiplying the seafloor spectrum by $[1 - z_0(k)/H(k)]^2$ [cf. Eq. (4)]. The virtual floor approach provides a nonlocal correction to the linear theory and thus should yield better estimates of the global energy flux distribution than the local correction in [26].

In the future the virtual floor approach should be extended to more realistic oceanic conditions by including three dimensionality [28,37–40], finite depth [17,21], and nonuniform stratification [41,42]. The virtual floor concept
could also be useful in the parametrization of internal wave breaking and dissipation in systems with multiple steep ridges [43–46], abyssal hills [26], rough topography [47], and internal tide scattering [48].

We thank Amadeus Dettner, Matthew Paoletti, and Bruce Rodenborn for discussions, and Neil Balmforth for providing the Green’s function data in Fig. 4(a). The research was supported by the Office of Naval Research MURI Grant No. N000141110701. L. Z. acknowledges the support of the 2013-14 F. V. Hunt Postdoctoral Research Fellowship in Acoustics. The computations were done at the Texas Advanced Computing Center.

1izhang@chaos.utexas.edu
swinney@chaos.utexas.edu

[34] Recall that the linear theory for weak topography predicts a power about half of that radiated by an isolated knife edge.